One Hyperbola

Time required
45 minutes

Teaching Goals:

1. Students interpret the given word problem and complete geometric constructions according to the condition of the problem.
2. Students choose an independent variable and define it as a constraint in the geometric construction.
3. Students analyze expressions for the optimized quantities, such as length, sum of lengths (perimeter), and area by doing the following:
   - Determine the domain of a function based on conditions of the problem
   - With the help of the software, visually determine if a quantity reaches a minimum or maximum.
   - Graph expression as a function of a chosen independent variable to determine the existence of a minimum or maximum
   - Determine expressions for the extrema of functions with the help of the software
   - Confirm solutions for extrema analytically

Prior Knowledge

- Students should know the properties of a rectangle.
- Students should know basic properties of an hyperbola, including its equation, graph, and asymptotes.
- Students should know that if the product of two positive quantities is constant, then the minimum of the sum of these quantities is reached when these two quantities are equal:

  If *a*b=constant*, then *a+b=min* when *a* = *b*; or geometrically, if the area of a rectangle is constant, the perimeter is minimal when it is a square.

Problem:

Given a branch of an hyperbola in the 1st quadrant defined by the equation $y = \frac{a}{x}$. Consider the open region bounded by the coordinate axes and the hyperbola, containing a rectangle with two sides on the coordinate axes and one vertex on the hyperbola.

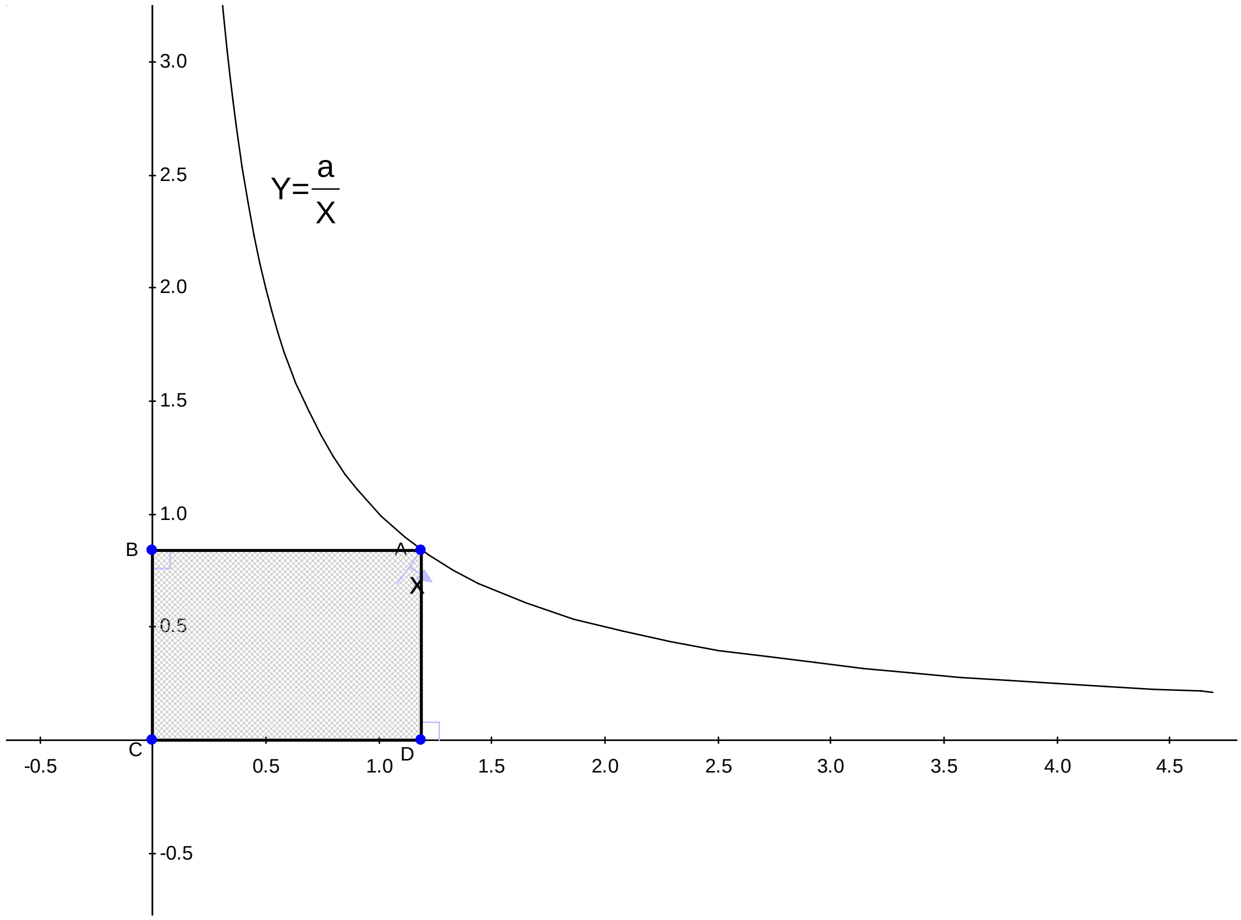
a. Investigate if the area of this rectangle can have a minimum or maximum value and justify your answer.
b. Find the extremal perimeter of this rectangle. Determine if this is a minimum or a maximum.

c. Find the extremal length of the diagonal of this rectangle. Determine if this is a minimum or a maximum

Part 1 – Investigating Area of the Rectangle

In this part of the problem students analyze the word problem and complete a geometric construction of the hyperbola and an inscribed rectangle according to the specifications of the problem. Students then choose an independent variable and define it in the construction. Students can use the dynamic features of the software to analyze the problem qualitatively first; then they can determine the symbolic expression for the perimeter of the rectangle using the symbolic calculation features of *Geometry Expressions* and analyze the problem graphically and analytically. Here are the steps of construction and questions.

1. Open a new file. If axes do not appear in the blank document, create them by clicking the **Toggle grid and axes** icon on the top toolbar.
2. Select **Function** from the **Draw** toolbox. In the **Function Type** dialog enter *a/x* for the Y value. The hyperbola will appear on the graph. If needed, adjust the window by using one of the scaling icons or drag the graph. Students only need to see the 1st quadrant of the coordinate plane
3. Select **Point** from the **Draw** toolbox and draw a point on the hyperbola in the 1st quadrant.
4. Select the point and the hyperbola. With both of them selected, choose **Point proportional along curve** from the **Constrain** toolbox. Enter x in the data entry box. Here x is the x-coordinate of the point on the parabola. This is an independent variable.
5. Select **Polygon** from the **Draw** toolbox and draw a rectangle with one vertex on the point on the hyperbola, one on the origin, and two on the coordinate axes.
6. Click the segment connecting the point on the hyperbola with the point on the Y axis and the Y axis outside the segment boundary. (If you try to constrain the top segment of the box perpendicular to the segment on the Y axis you will receive an over constrained message, so make the constraint with the axis itself, not the segment lying on the axis.) With both of them selected choose **Perpendicular** from the **Constrain** toolbox. Repeat for the segment connecting the hyperbola to the X axis.

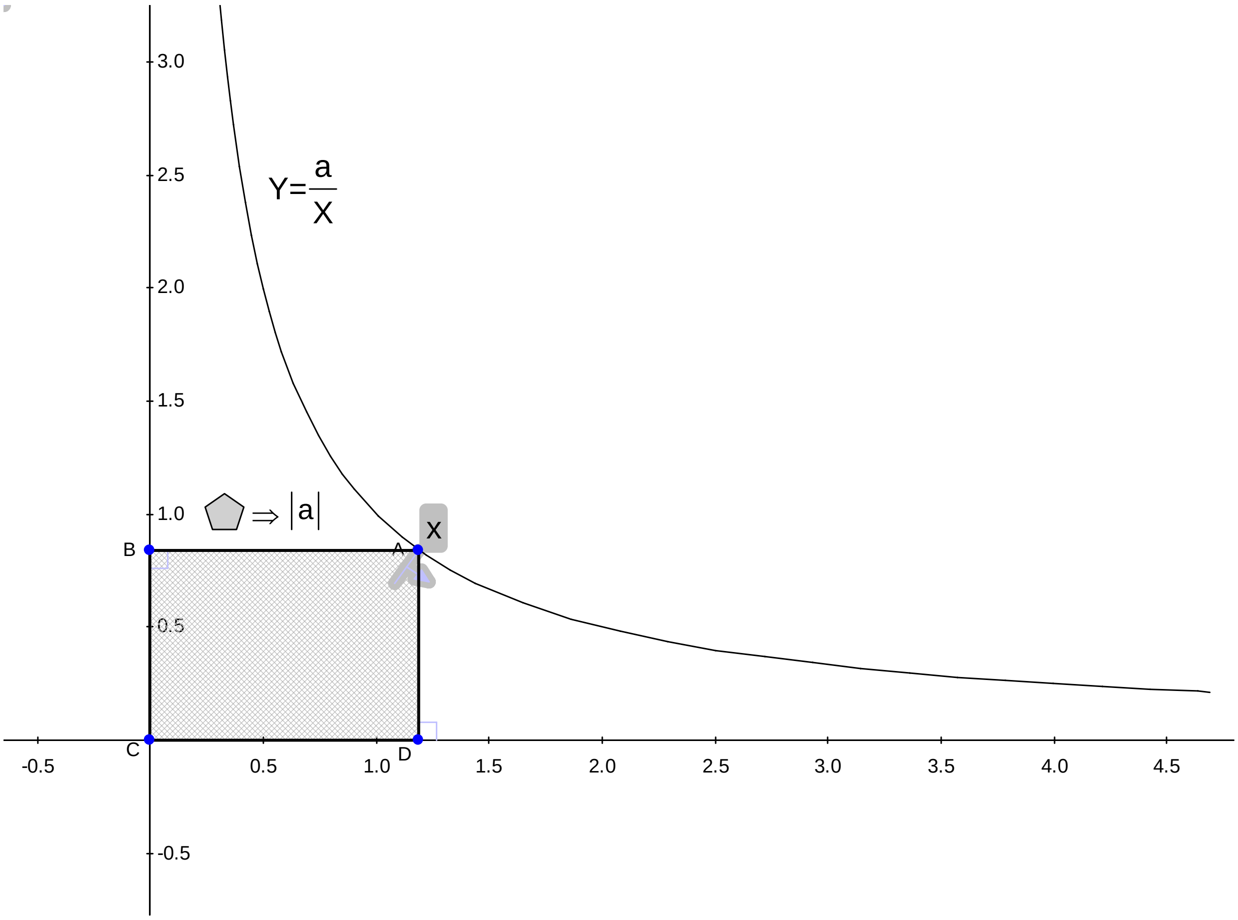


Q1. Drag the point on parabola and observe the changes in the area of the rectangle. Do you think the area has a minimum? A maximum?

A. From the observations students may not conclude an exact answer, since as one side of the rectangle increases, another side decreases. The correct answer is that the area of this rectangle remains constant and does not depend on the position of the point on the hyperbola.

Q2. Find an expression for the area of the rectangle. Explain your work

A. The area of the rectangle can be found as $A = X \cdot Y = X \cdot \frac{a}{X} = a = const$. Students can confirm this calculation using the **Area** tool from the **Calculate** toolbox.



Note: it is important for students to be exposed to optimization problems with functions that do not necessarily have extremal values, such as this problem.

Part 2 – Optimizing Perimeter of the Rectangle

In this part of the problem students explore how the perimeter changes using dynamic features of the software first, then they can determine the symbolic expression for the perimeter of the rectangle and analyze its function analytically and graphically. Use the constructed rectangle in Part 1 to answer these questions.

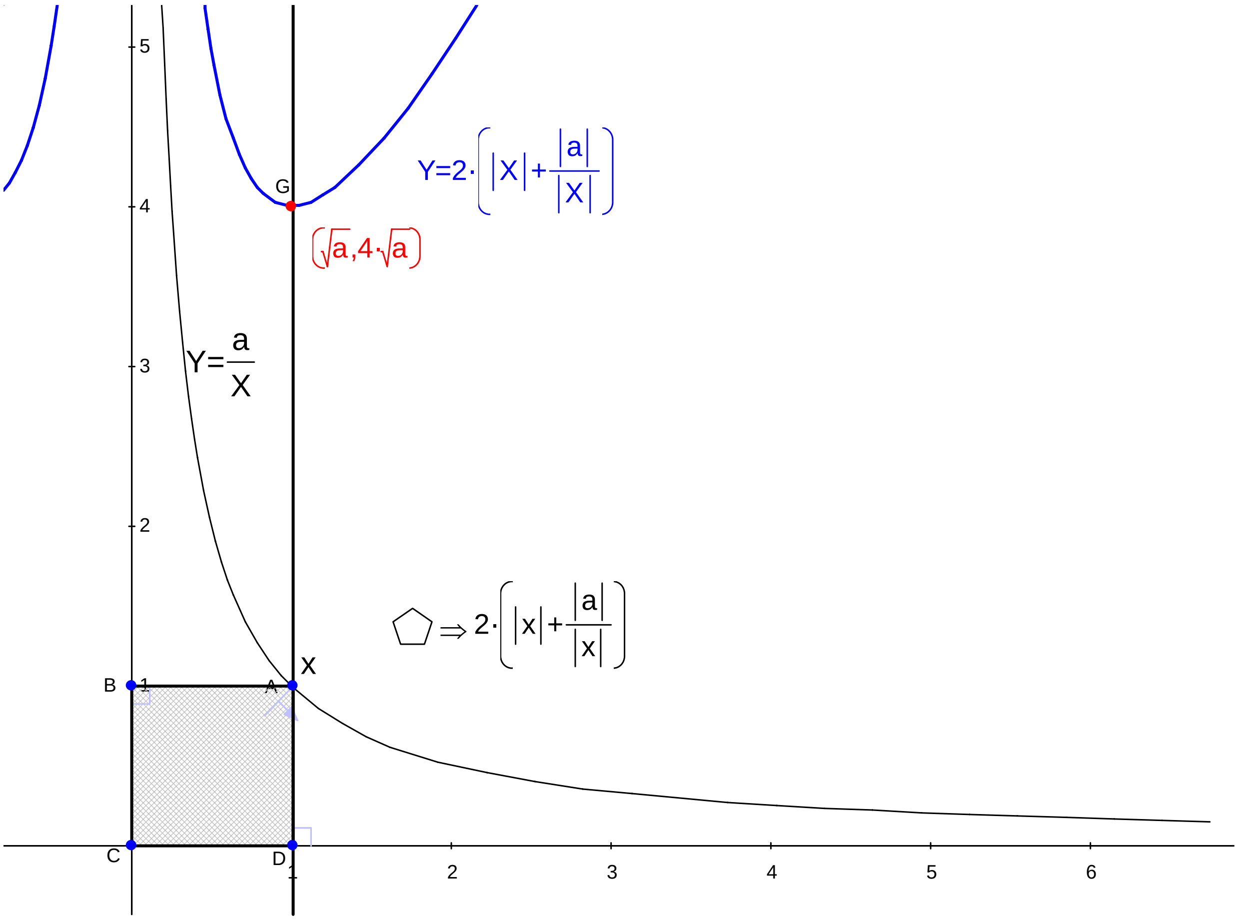
Q1. Drag the point on the hyperbola and observe the changes in the perimeter of the rectangle. Do you think the perimeter has a minimum? A maximum?

A. From their observations students may conclude that when the point on the hyperbola is approaching either of the coordinate axes, the perimeter is getting infinitely large, so there is no maximum value. However, that means there should be a minimum value. From ideas of symmetry of hyperbola they can then conjecture that the minimum of the perimeter will occur when the rectangle becomes a square, and the vertex of the rectangle that is on the hyperbola is located on the vertex of the hyperbola.

Q2. Find an expression for the perimeter algebraically. Then check your work using the software. Plot this expression as a function of x and investigate if this function has a minimum value.

A. The perimeter can be found as $P = 2(x+y) = 2\left(x + \frac{a}{x}\right)$. Students can find the expression for the perimeter using the software to confirm their answer. Using the fact that the area of the rectangle is constant, students can conclude that the minimal perimeter will be achieved when the rectangle becomes a square. Thus, the perimeter reaches its minimum when $x = \frac{a}{x}$, $x = \sqrt{a}$. So the minimum value of the perimeter is $P = 4\sqrt{a}$. Students then graph this function and determine that the point they found is indeed the minimum of the perimeter. Here are the steps of finding the expression for the perimeter and constructing the graph.

1. Click the rectangle and choose **Perimete**r from the **Calculate** toolbox, **Symbolic** tab. The software will produce an expression for the perimeter in terms of a and x.
2. Right click the expression for the perimeter, choose **Copy As/ String**.
3. Select **Function** from the **Draw** toolbox. In the **Function Type** dialog, Cartesian Type, paste the expression into the Y= line. Press OK. The graph of the function will appear on the screen. If you can't see the function, use one of the scaling options to see the plot of the function.
4. Select **Point** from the **Draw** toolbox and draw a point anywhere in the blank space. Select the point and choose **Coordinate** tool from the **Constrain** toolbox. Type $sqrt(a)$ for x_0 and $4*sqrt(a)$ for y_0. Observe the point jumping to the function minimum.
5. Select both, point A and the x-axis. Choose **Perpendicular** from the **Construct** toolbox. A vertical line will appear on the graph.
6. Move point *A* to the minimum and observe that the rectangle becomes a square.



Part 3 – Optimization of the Diagonal of the Rectangle.

In this part of the problem students explore how the diagonal of the rectangle (with endpoints *B* and *D* above) changes using the dynamic features of the software. Then they can determine a symbolic expression for the diagonal of the rectangle and analyze the function graphically and analytically. Note: since the diagonals of the rectangle are equal, it is the same problem to consider either diagonal; however, by choosing diagonal *BD* it may be more obvious for students how the length of this diagonal changes with the movement of point *X* along the hyperbola. The teacher may ask students to save their existing file that includes constructions and computations for the rectangle, and delete all unnecessary constructions and calculations. If the teacher decides to start with a blank file, repeat steps 1 – 6 from ***Part 1***. Here are the steps on of construction and questions from this point forward.

Select **Line Segment** tool from the **Draw** toolbox. Draw a segment BD (connecting vertices on the coordinate axes).

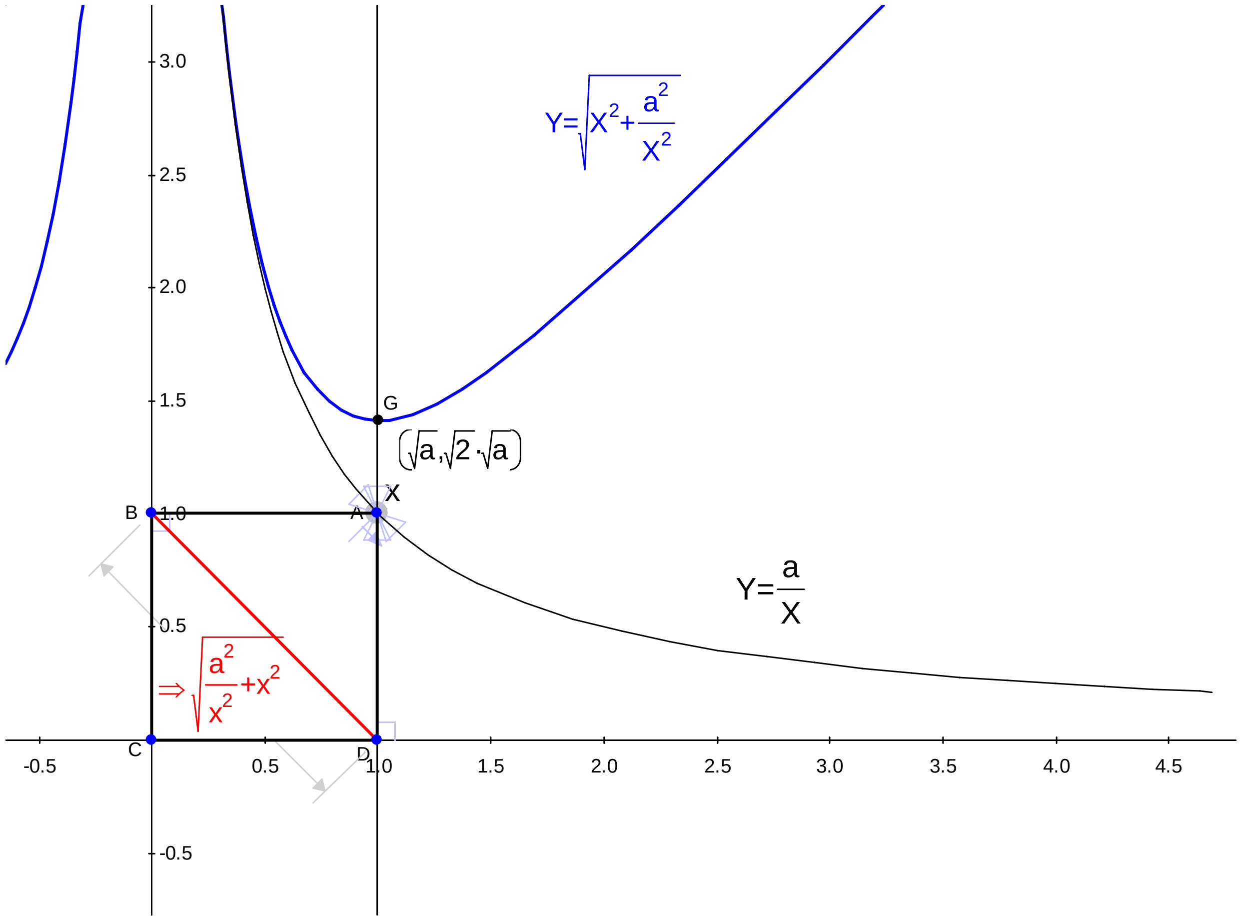
Q1. Drag point X along the hyperbola and observe the changes in the length of the diagonal of the rectangle. Do you think the diagonal has minimum length? A maximum length?

A. From the observations students may conclude that when the point on the hyperbola approaches either of the coordinate axes, the diagonal becomes infinitely long, so there is no maximum value. However, that means there should be minimum value. From ideas of symmetry of hyperbolas they can then conjecture that the minimum of the diagonal length will occur when the rectangle becomes a square, and the vertex of the rectangle that is on the hyperbola is located in the vertex of the hyperbola.

Q2. Find an expression for the length of the diagonal algebraically. Then confirm your expression using software. Plot this expression as a function of x and investigate if this function has minimum value.

A. The length of the diagonal can be found as $L=\sqrt{x^2+y^2}=\sqrt{x^2+\left(\frac{a}{x}\right)^2}$. Students can find the expression for the diagonal using the software to confirm their equation. Using the fact that area of the rectangle is constant, it also means that the square of the area is constant. Therefore the sum of squares of the sides is minimal when the rectangle becomes a square. Thus, the length of the diagonal is minimum when $x=\frac{a}{x}, x=\sqrt{a}$. So the minimum value of the diagonal is $L=\sqrt{a+a}=\sqrt{2a}$. Students then graph this function and determine that the point they found is indeed the minimum of the perimeter. Here are the steps of finding the expression for the diagonal and constructing the graph.

1. Click the diagonal and choose **Distance/Length** from the **Calculate** toolbox, **Symbolic** tab. The software will produce an expression for the diagonal in terms of a and x.
2. Right click the expression for the perimeter, choose **Copy As / String**.
3. Select **Function** from the **Draw** toolbox. . In the **Function Type** dialog, Cartesian Type, paste the expression into the Y= line and click OK. The graph of the function will appear on the screen. If you can't see the function, choose one of the scaling options to adjust the plot.
4. Select **Point** from the **Draw** toolbox and draw a point anywhere in the blank space. Select the point and choose **Coordinate** from the **Constrain** toolbox. Type $sqrt(a)$ for x_0 and $sqrt(2*a)$ for y_0. Observe the point jumping to the function minimum.
5. Select both, point *A* and the x-axis. Choose **Perpendicular** from the **Construct** toolbox. The vertical line will appear on the graph.
6. Move point *A* to its minimum and observe that the rectangle becomes a square.



$Y=\sqrt{X^2+\frac{a^2}{X^2}}$
G
$(\sqrt{a},\sqrt{2}\cdot\sqrt{a})$
X
B
A
C
D
$Y=\frac{a}{X}$
$\Rightarrow\sqrt{\frac{a^2}{x^2}+x^2}$
3.0
2.5
2.0
1.5
1.0
0.5
-0.5
-0.5
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
4.5