Trammels

Archimedes Trammel is a device consisting of a handle attached at two pivot points to sliders which can slide to and fro in perpendicular slots (figure 1). In this activity, we will investigate the curve formed by the trammel.

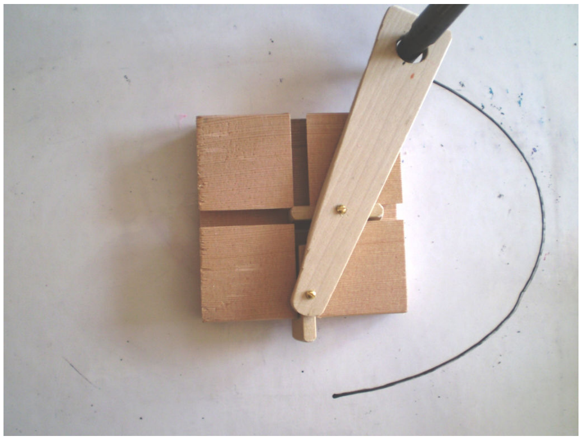

Figure 1: Archimedes trammel

We can model the trammel simply in Geometry expressions by drawing a line segment AB such that A is snapped to the y-axis, and B is free. We then create a point C at the intersection of AB with the x-axis. The model is further constrained by specifying the length AC to be a, and the length AB to be b (figure 2).

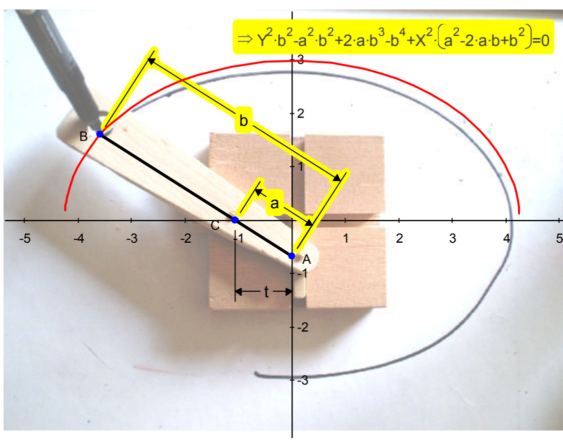

Figure 2: Geometry Expressions model of the trammel computes the equation of the curve and verifies that it is an ellipse

In order to create a curve, we need to specify a parameter of the resulting motion. A suitable parameter is the distance between the point C and the y-axis. In figure 2, we have specified this distance to be t.

Create the curve by selecting point B then choosing the locus tool, and selecting t as the parameter.

Having created the curve, you can select it and ask for its implicit equation.

Examination of the equation shows that it has terms in X^2, Y^2 and a constant term. It is therefore in the correct form for an ellipse (or hyperbola, but I think it is visually clear that it is an ellipse).

Given that this is an ellipse, can you work out the lengths of its major and minor axes?

The major axis length is clear from figure 2. To compute the minor axis, you could try this starting from the equation, setting X=0 and solving for Y, or perhaps you can work it out directly from the geometry by looking at the trammel picture in figure 2. The answer is seen in figure 3.

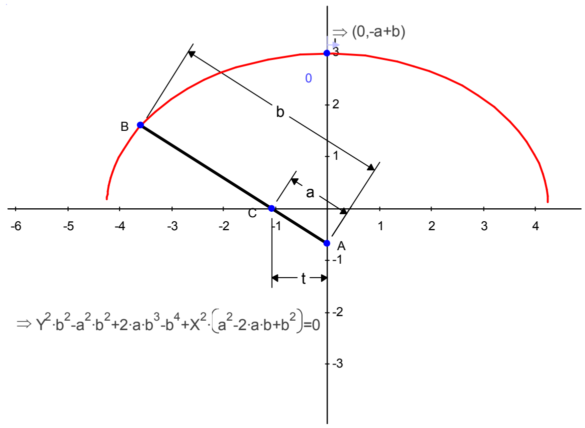

Figure 3: Minor axis of the ellipse (location of point at parametric location 0 on the curve)

Ellipse Foci

An alternative method of drawing an ellipse is to pin a piece of string at two points (called foci), and trace round keeping the string taught. Where would we need to put the foci and what length string would we need to trace out the same ellipse as the trammel?

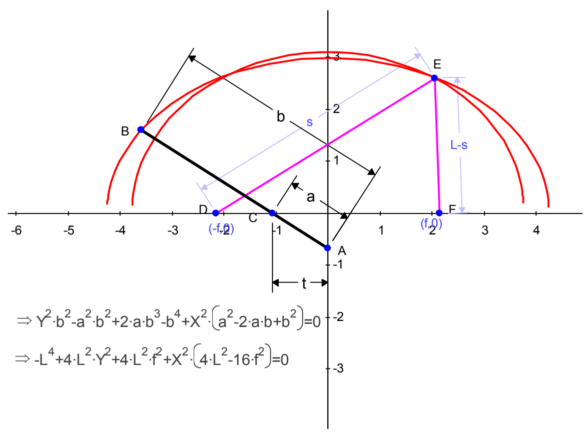

Figure 4: Ellipse formed using a string length L pinned at (f,0) and (-f,0)

Figure 4 shows a general arrangement where the foci are at point (f,0) and (-f,0), and the string is length L.

You can try dragging the foci and the point E to match the trammel curve. As there are two variables, f and L, this task is not easy. If we could fix one of the parameters, then drag the other the problem would be simpler.

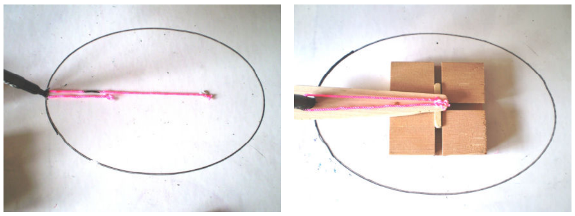

Figure 5: As the two foci are equidistant from the center of the ellipse, the string length should be twice the length of the trammel handle

If we imagine aligning the string with the ellipse's major axis (figure 5), we can observe that the string length must be twice the length of the handle. (Imagine pulling the foci gradually towards the center, while keeping the string taught with the pen). Figure 6 shows the result of setting the string length to be 2b.

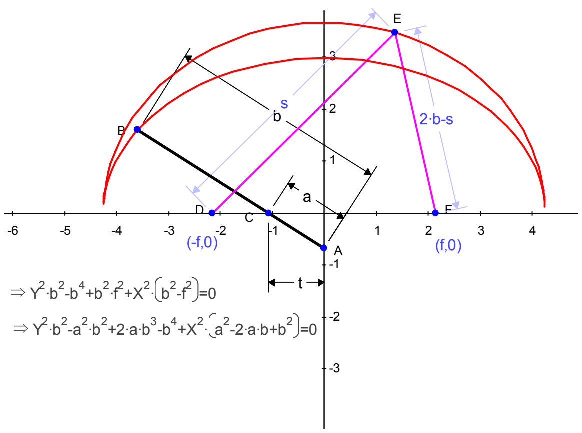

Figure 6: Setting the string length to 2b

It is now easy to drag a focus to line up the curves, but can we work out an algebraic expression for f in terms of a and b?

We could examine the equations of the curves. The coefficients of Y^2 are equal, what value of f would make the coefficients of X^2 equal?

Alternatively we could think geometrically. Looking at figure 7, can you imagine a right angled triangle which is formed when the pen passes through the minor axis of the ellipse? We already know the length of the vertical side of this triangle (fig. 3). What is its hypotenuse? What is the length of its horizontal side?

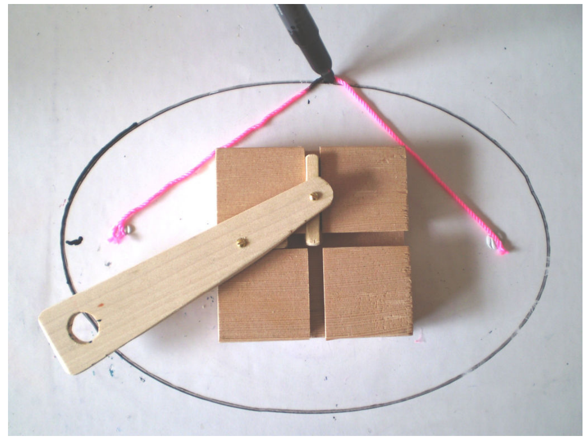

Figure 7: The string stretched approximately through the minor axis of the ellipse

Either algebraically or geometrically, we should derive the solution $f = \pm\sqrt{2ab - a^2}$. Figure 8 shows the result of using these foci, and we observe that the curves align geometrically and have the same algebraic equation.

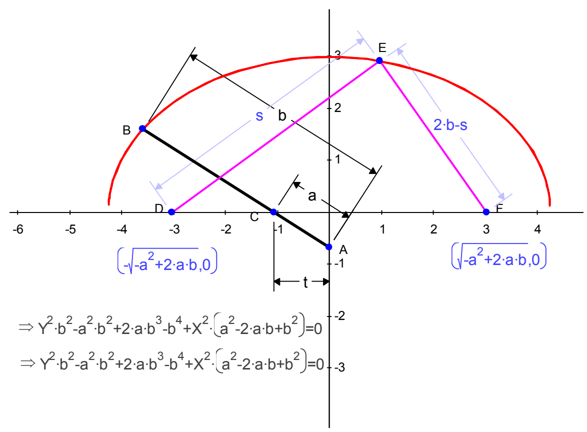

Figure 8: Foci at $x = \pm\sqrt{2ab - a^2}$ aligns the two curves

For further thought

Figure 9 shows a trammel with 3 slots. Why is it surprising that this moves? What curve does the handle trace out? Why?

Figure 9: A trammel with 3 slots.