

Translation Along Coordinate Axes

Time required
45 minutes

Teaching Goals:

1. Students verify that the graph of the function $y = f(x) + b$ is derived from the graph of the function $y = f(x)$ by translation along the y-axis by the vector $(0, b)$.
2. Students verify that the graph of the function $y = f(x - a)$ is derived from the graph of the function $y = f(x)$ by translation along the x-axis by the vector $(a, 0)$.
3. Students verify the commutative property of horizontal and vertical translations of the graph of a function.
4. Students determine that two translations by vectors $(a, 0)$ and $(0, b)$ is equivalent to the translation by the vector (a, b) .
5. Students determine that the graph of the function $y = f(x - a) + b$ is derived from the graph of function $y = f(x)$ by translation by the vector (a, b) .
6. Students determine the position of the translated graph of a function depending on the sign of parameters a and b .
7. Students verify these results with the help of the software.

Prior Knowledge

- Students should know the definition of a vector.
 - Students should know the coordinate representation of vectors.
 - Students should know how to add vectors in vector form and in coordinate form.
 - Students should know the coordinate form of translation by a vector, including special cases along each coordinate axis.
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Problem:

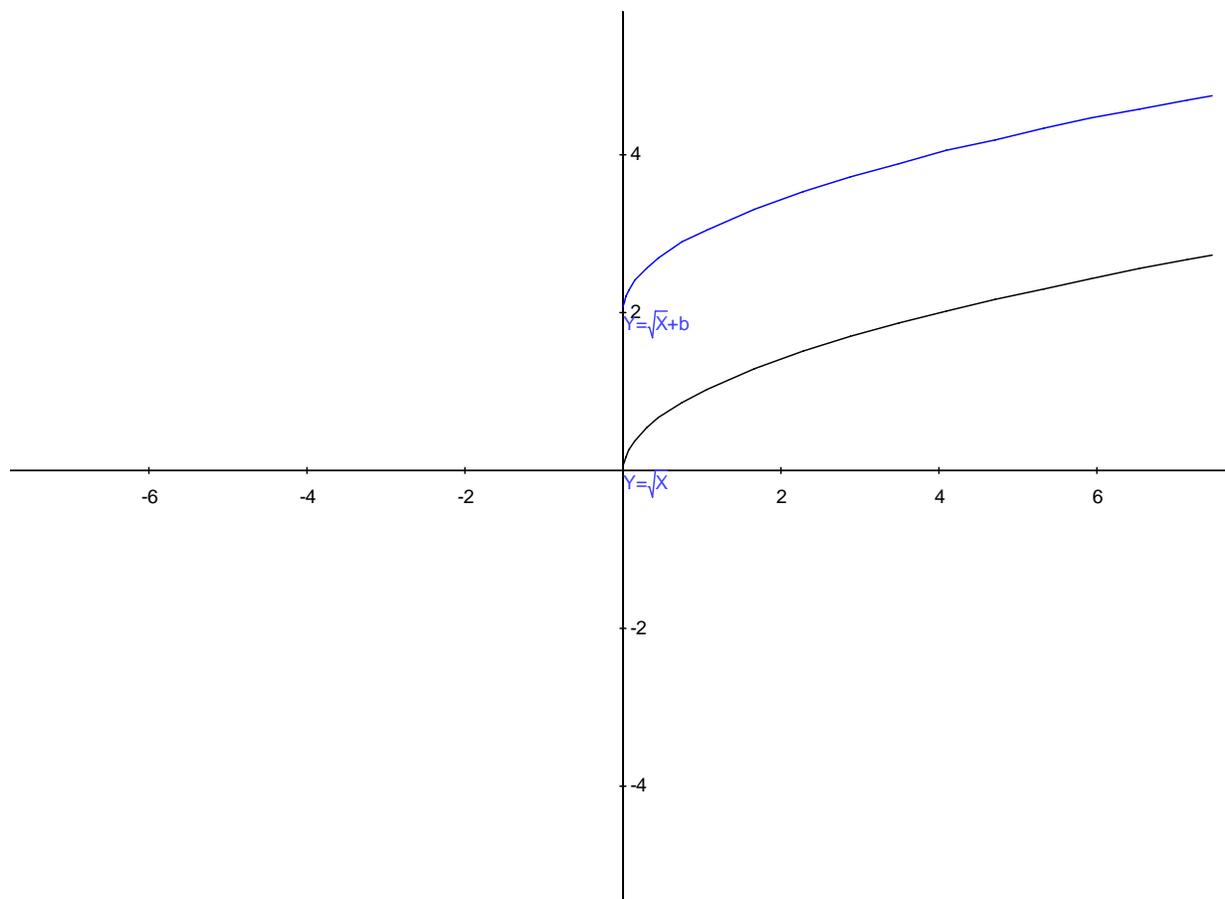
Given the function $y = \sqrt{x}$, investigate translation of the graph of the function.

Part 1 – Translation Along the Y-Axis

In this part of the problem students investigate vertical translations and discover that the graph of the function $y = f(x) + b$ is achieved by translating the graph of the function $y = f(x)$ by vector $(0, b)$ for any function $f(x)$. They start working with the graph of the function $y = \sqrt{x}$ and then generalize for an arbitrary $f(x)$. Here are the steps of construction and questions.

1. Open a new file. If axes do not appear in the blank document, create them by clicking the **Toggle grid and axes** icon on the top toolbar.

2. Select **Function** from the **Draw** toolbox. Type: $\text{sqrt}(X)$ for the function.
3. Draw a second function and enter: $\text{sqrt}(X)+b$ for the function. Select the curve, right-click and select **Properties** from the context menu to change the line color to Blue.



Q1. As you drag the graph of function $y = \sqrt{x} + b$ up and down along the y -axis, what do you observe about the sign of parameter b with respect to the position of this graph relative to the graph $y = \sqrt{x}$?

A: It is expected that students will make the following conjecture: $b > 0$ when the graph of $y = \sqrt{x} + b$ is above the graph of $y = \sqrt{x}$; and $b < 0$ when the graph of $y = \sqrt{x} + b$ is below the graph of $y = \sqrt{x}$.

In order to see that, students can observe the value of b in the **Variables** toolbox while dragging the graph of the function $y = \sqrt{x} + b$.

Q2. What is the geometric transformation from the graph of $y = \sqrt{x}$ to $y = \sqrt{x} + b$.

A. It is expected that students will bring up translation by a vertical vector.

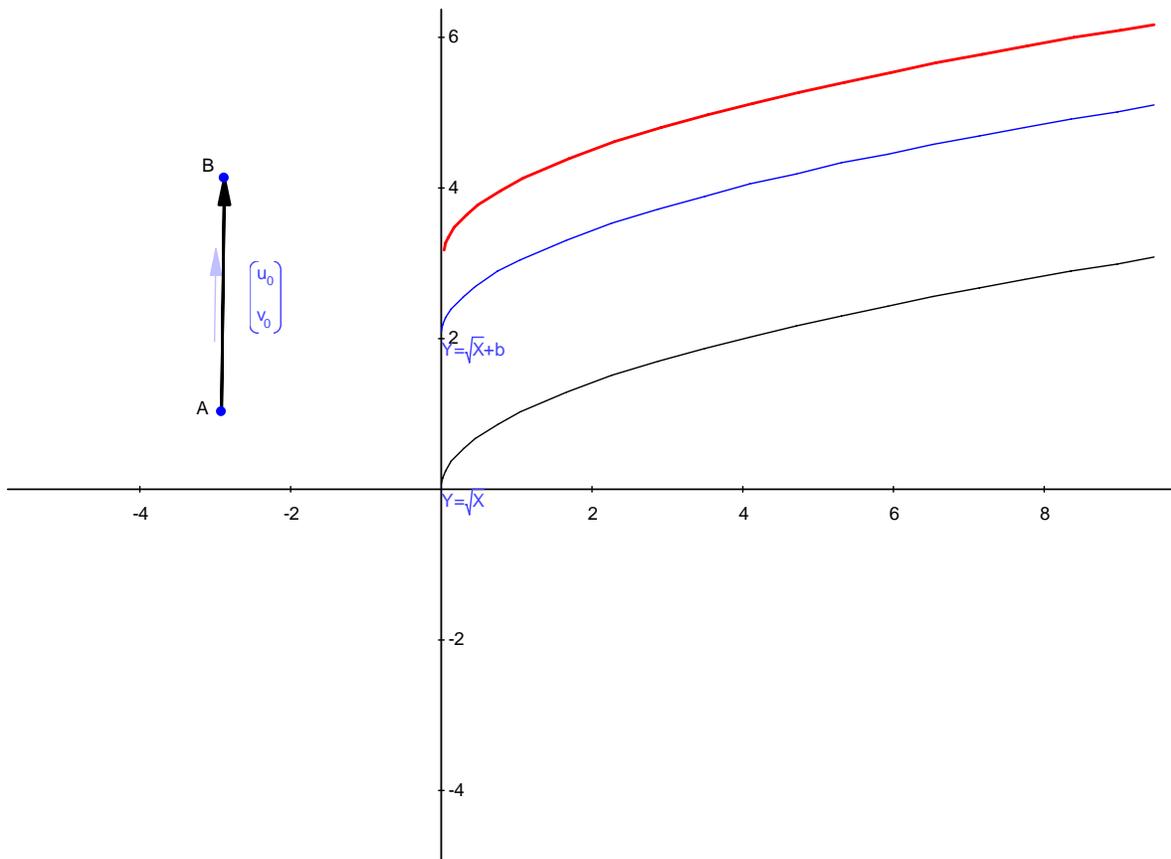
In order to verify this with the software, follow these steps.

1. Select **Vector** from the **Draw** toolbox. Construct a vector AB anywhere on the plane. Click the selection arrow.
2. Click the graph of the function $y = \sqrt{x}$. Select **Translation** from the **Construct** toolbox, select vector AB and the translation of the graph will appear on the screen.
3. Select the translated graph, right-click, select **Properties** from the context menu, and choose Line Style – Solid 2, Line Color – Red.

Q3. What are the coordinates of the vector of translation that transforms graph of $y = \sqrt{x}$ to the graph of $y = \sqrt{x} + b$? Make a conjecture and verify it with the help of the software.

A. The teacher may help students come up with a conjecture by pointing to the characteristic points of the graph, in this case point (0, 0) which is the vertex of the semi-parabola. This hint should help them determine the coordinates of the vector of translation. Students will also use the software to help find the answer and verify their conjecture.

1. Select vector AB, choose **Coefficients** from the **Constrain** toolbox. The coefficients $\begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$ of the vector will appear in the edit box. Accept the default values by pressing *Enter*.



2. By moving the endpoint of the vector B students can overlap the translated image with the

graph of the function $y = \sqrt{x} + b$ and observe the values of u_0 , v_0 , and b in the **Variables** toolbox to answer the question and to verify their conjecture.

The expected answer: $u_0 = 0$, $v_0 = b$.

3. In order to verify this, students can double click the coefficients and substitute these values. The translated image will overlap with the graph of the function $y = \sqrt{x} + b$. Drag point B to confirm that.

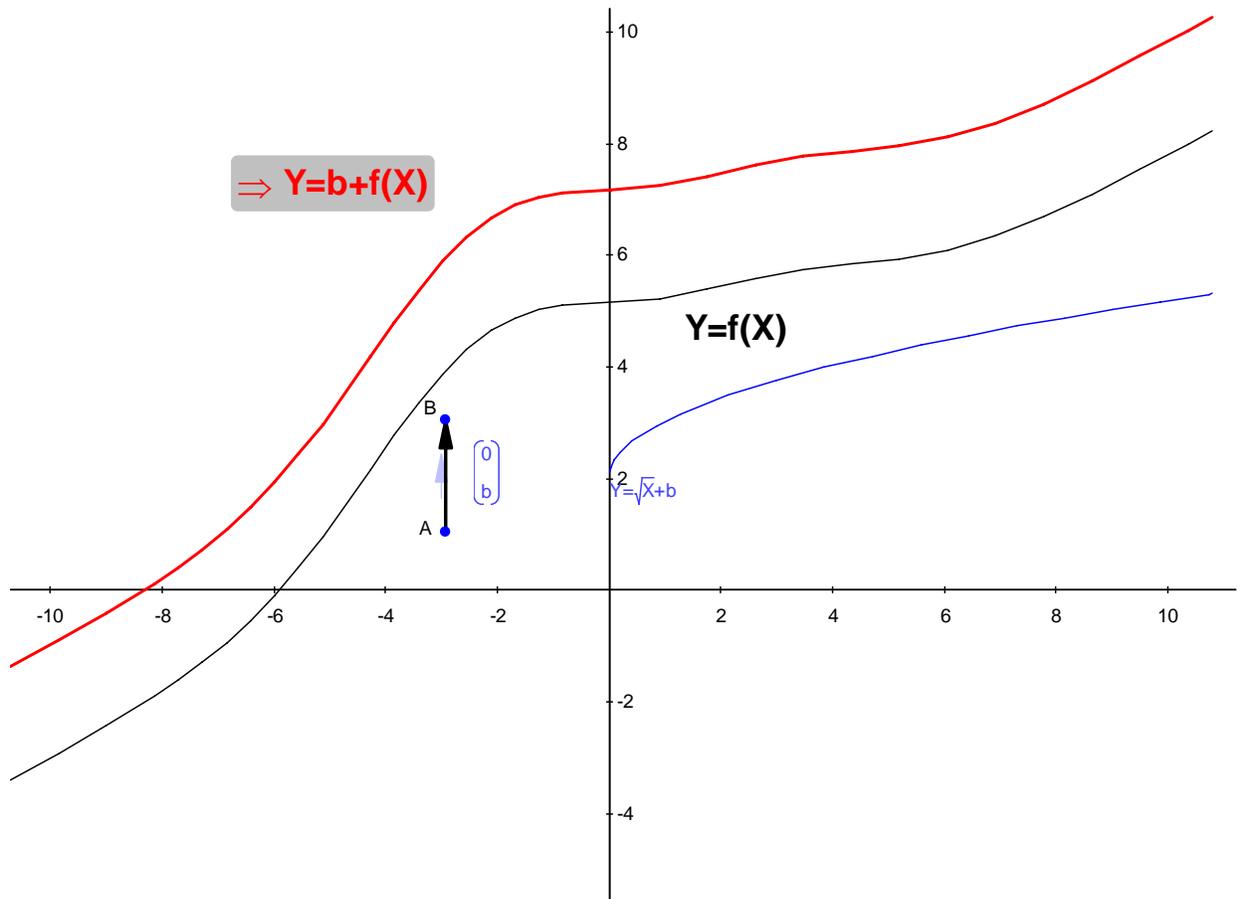
Q4. What vector will translate the graph of function $y = \sqrt{x} + b$ back to the graph of function $y = \sqrt{x}$? Make a conjecture and verify with the software.

A. The vector of translation is $(0, -b)$.

Q5. Generalize these results for an arbitrary function $f(x)$ and confirm your conclusion with the help of the software.

A. Students should conclude that the graph of function $y = f(x) + b$ is derived from graph of function $y = f(x)$ by translation along the y -axis by the vector $(0, b)$. They should also conclude that the graph of function $y = f(x)$ is derived from graph of function $y = f(x) + b$ by translation along the y -axis by the vector $(0, -b)$. They can verify this by editing the equation of the function:

1. Double click the equation of the function and change it to: $f(x)$. The graph of the arbitrary function will appear on the screen.
2. Click the image of the translated function and choose **Implicit Equation** from the **Calculate** toolbox. The equation $y = b + f(x)$ will appear on the screen.

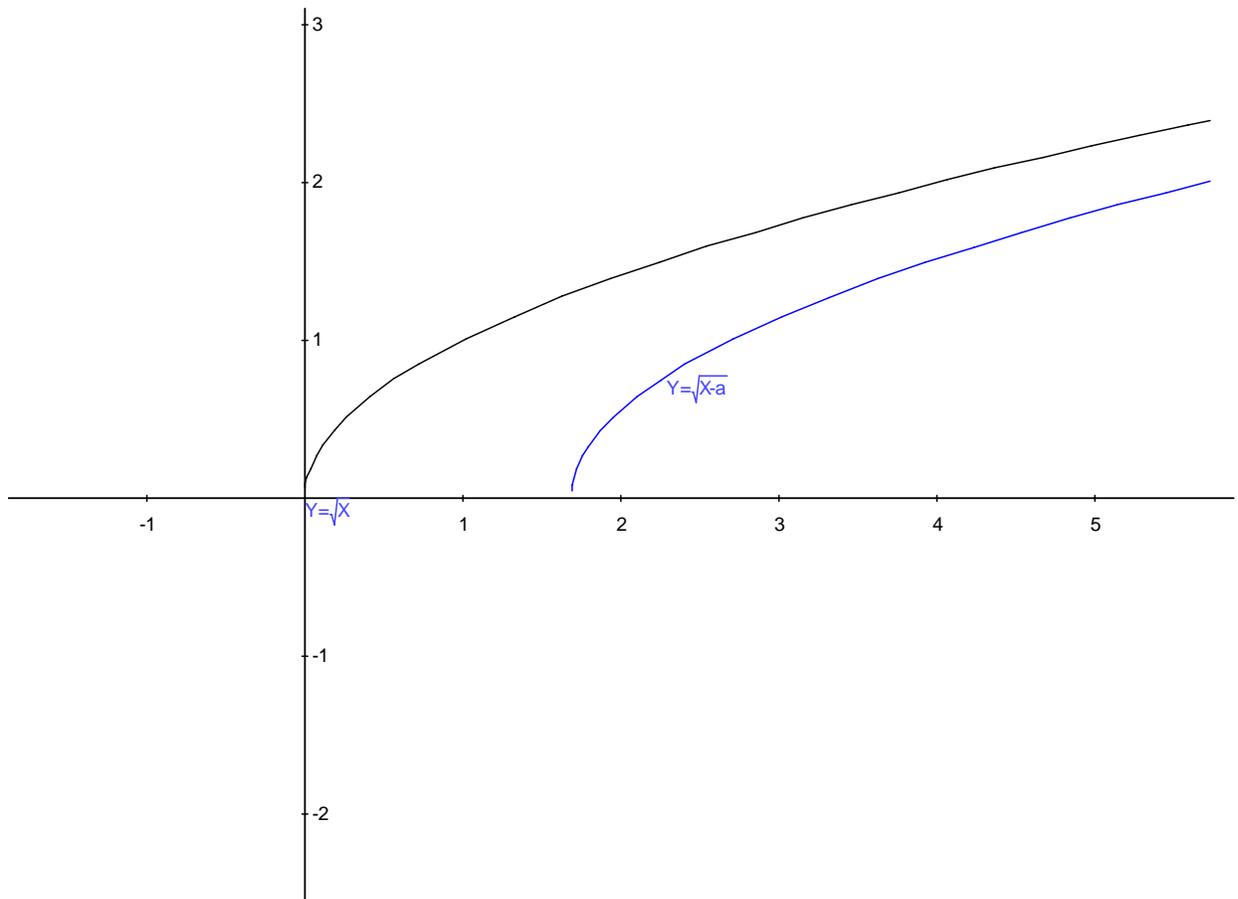


Similarly, students can verify inverse translation with the help of the software.

Part 2 – Translation Along the X-Axis

In this part of the problem students investigate horizontal translations and discover that the graph of function $y = f(x - a)$ is always derived by translation from the graph of the function $y = f(x)$ by the vector $(a, 0)$ for any function $f(x)$. They start working with the graph of the function $y = \sqrt{x}$ and then generalize for an arbitrary $f(x)$. Here are the steps of construction and questions.

1. Open a new file. If axes do not appear in the blank document, create them by clicking the **Toggle grid and axes** icon on the top toolbar.
2. Select **Function** from the **Draw** toolbox. Type: $\text{sqrt}(X)$ for the function.
3. Select **Function** from the **Draw** toolbox. Type: $\text{sqrt}(X - a)$ for the function. Select the function, right-click and select **Properties** from the context menu to change the line color to Blue.



Q1. As you drag the graph of the function $y = \sqrt{x-a}$ left and right along the x-axis, what do you observe about the sign of parameter a with respect to the position of this graph relative to the graph $y = \sqrt{x}$?

A: It is expected that students will make the following conjecture: if $a > 0$, the graph of $y = \sqrt{x-a}$ is shifted to the right of the graph of $y = \sqrt{x}$; if $a < 0$, the graph of $y = \sqrt{x-a}$ is shifted to the left of the graph of $y = \sqrt{x}$.

In order to see this, students can observe the value of a in the **Variables** toolbox while dragging the graph of the function $y = \sqrt{x-a}$.

Q2. What is the geometric transformation from the graph of $y = \sqrt{x}$ to $y = \sqrt{x-a}$?

A. It is expected that students will bring up translation by a horizontal vector.

In order to verify this with the software, follow these steps:

1. Select **Vector** from the **Draw** toolbox. Construct a vector AB anywhere on the plane. Click the selection arrow.
2. Click the graph of the function $y = \sqrt{x}$. Select **Translation** from the **Construct** toolbox,

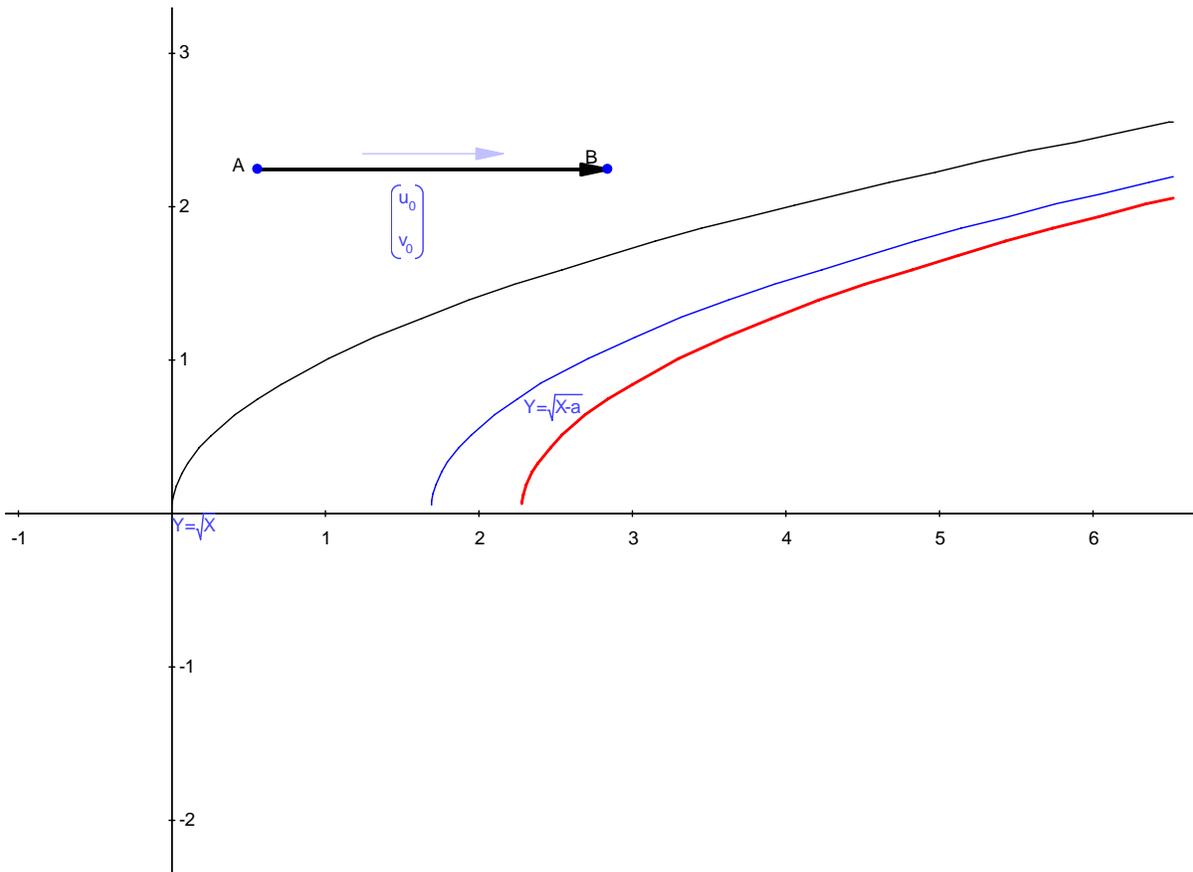
select vector AB and the image of the graph will appear on the screen.

3. Select the translated image, right-click and select **Properties** from the context menu to make Line Style – Solid 2, and Line Color – Red.

Q3. What are the coordinates of the vector of translation that transform the graph of $y = \sqrt{x}$ to the graph of $y = \sqrt{x-a}$? Make a conjecture and verify it with the help of the software.

A. The teacher may help students come up with a conjecture by pointing to the characteristic points of the graph, in this case point (0, 0) which is the vertex of the semi-parabola. This hint should help them determine the coordinates of the vector of translation. Students will also use the software to help find the answer and verify their conjecture.

1. Select vector AB, choose **Coefficients** from the **Constrain** toolbox. The coefficients $\begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$ of the vector will appear in the edit box. Accept the default values by pressing *Enter*.



2. By moving the endpoint of the vector B students can overlap the translated image with the graph of the function $y = \sqrt{x-a}$ and observe the values of u_0 , v_0 , and a in the **Variables** toolbox to answer the question and to verify their conjecture.

The expected answer: $u_0 = a$, $v_0 = 0$.

3. In order to verify this, students can double click the coefficients and substitute these values. The translated image will overlap with the graph of the function $y = \sqrt{x - a}$. Drag point B to confirm this.

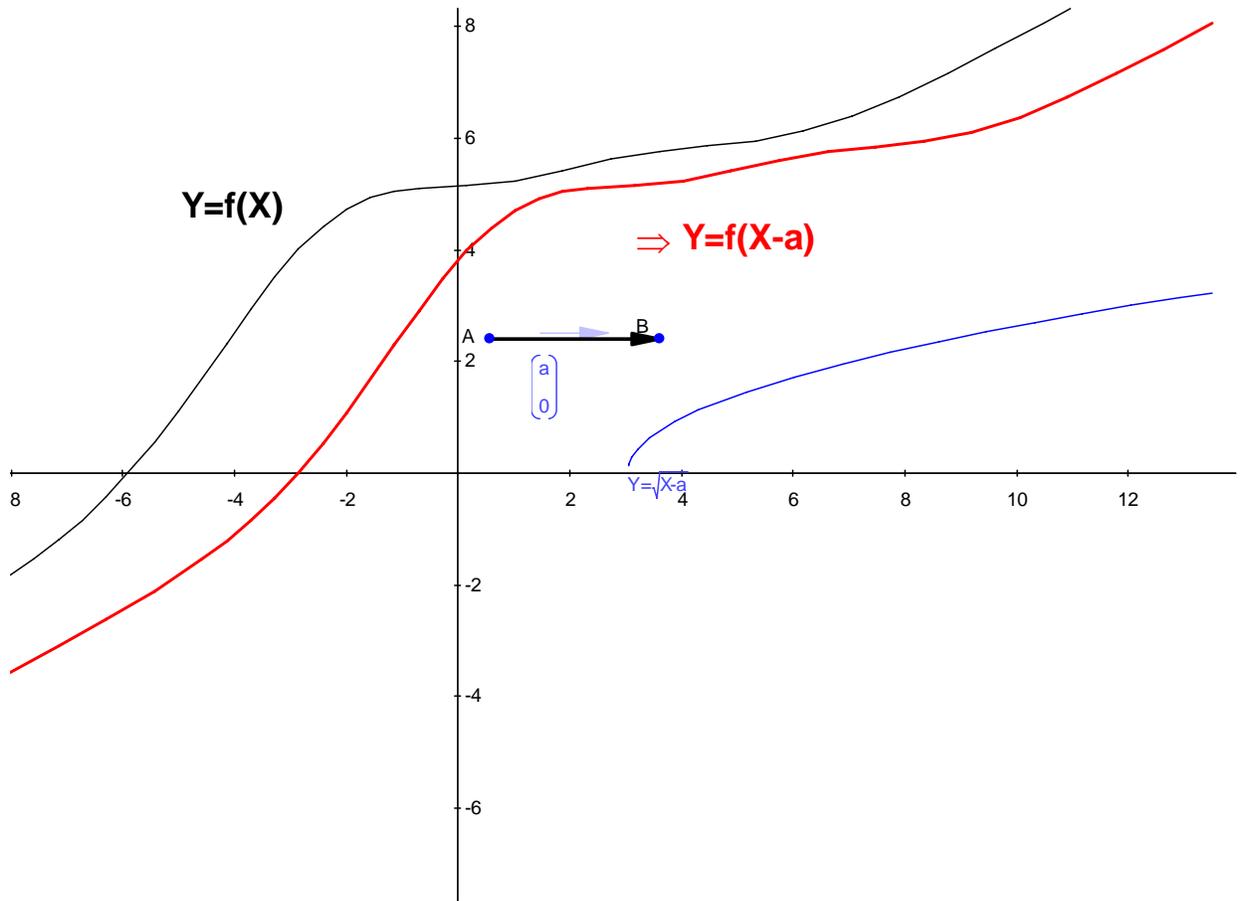
Q4. What vector will translate the graph of the function $y = \sqrt{x - a}$ back to the graph of the function $y = \sqrt{x}$? Make a conjecture and verify with the software.

A. The vector of translation is $(-a, 0)$.

Q5. Generalize these results for an arbitrary function $f(x)$ and confirm your conclusion with the help of the software.

A. Students should conclude that the graph of function $y = f(x - a)$ is derived from the graph of function $y = f(x)$ by translation along the x-axis by a vector $(a, 0)$. They should also conclude that the graph of function $y = f(x)$ is derived from the graph of function $y = f(x - a)$ by translation along the x-axis by a vector $(-a, 0)$. They can verify this by editing the equation of the function:

1. Double click the equation of the function and change it to: $f(x)$. The graph of the arbitrary function will appear on the screen
2. Click the image of the translated function and choose **Implicit Equation** from the **Calculate** toolbox. The equation $y = f(x - a)$ will appear on the screen.



Similarly, students can verify the inverse translation with the help of the software

Part 3 – Commutative Property of Translation

In this part of the problem students will investigate the commutative property of translations given by vectors $(0, b)$ and $(a, 0)$ along the coordinate axes of the graph of an arbitrary function $y = f(x)$.

Here are the steps of construction and questions.

1. Open a new file. If axes do not appear in the blank document, create them by clicking the **Toggle grid and axes** icon on the top toolbar.
2. Select **Function** from the **Draw** toolbox. Type $f(X)$ for the function.
3. Select **Vector** from the **Draw** toolbox. Construct two vectors, AB and CD. Click the selection arrow. Select vector AB, choose **Coefficients** from the **Constrain** toolbox, type: $(0, b)$ for the coefficients and press *Enter*. Select vector CD, choose **Coefficients** from the **Constrain** toolbox, type: $(a, 0)$ for the coefficients and press *Enter*.

Q1. Make a conjecture about relative position of the curves transformed by two methods 1) translate the graph of the function $y = f(x)$ by vector AB and then translate the resulting graph by the vector CD, or 2) translate the graph of the function $y = f(x)$ by vector CD and then translate the resulting graph by the vector AB. Verify your conjecture with the help of the

software.

A. Students will formulate their conjecture prior to working on the computer, then complete constructions on the computer and determine coincidence of the graphs, confirming or rejecting their conjectures. Here are the steps of the constructions.

At this point, it's useful to adjust the default Curve setting to the thinnest black line so that you can see when two curves are the same. From the drop-down **Edit** menu at the top of the screen select **Settings** and the **Geometry** tab. In the Curve box check that the Line Color is Black and the Line Style is Solid 1.

Method 1:

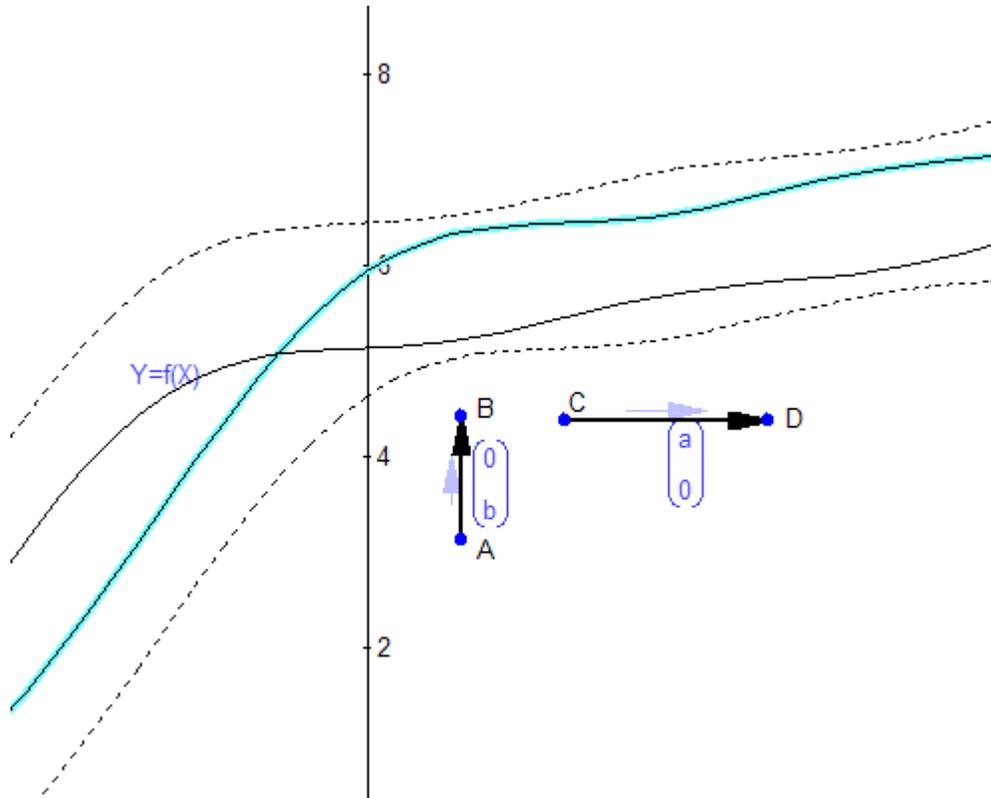
1. Click the graph of the function $y = f(x)$, select **Translation** from the **Construct** toolbox, then click vector AB. Click the resulting image, right-click and select **Properties** from the context menu to change the line style to Dot.
2. Click the translated image, select **Translation** from the **Construct** toolbox. Then click the vector CD. Click the translated image, right-click and select **Properties** from the context menu and choose Line Style – Solid 4, Line Color – Light Blue (or any pastel shade).

Method 2:

3. Click the graph of the function $y = f(x)$ and select **Translation** from the **Construct** toolbox, then click vector CD. Click the resulting image, right-click and select **Properties** from the context menu to change the line style to Dot.
4. With the graph still selected, choose **Translation** from the **Construct** toolbox. Then click vector AB.

Observe the image constructed by method 2 coincides with the image constructed by method 1.

Students should drag point B up and down and point D left and right to verify that this conjecture works for vectors of various lengths and directions.



Q2. What is the equation of this curve?

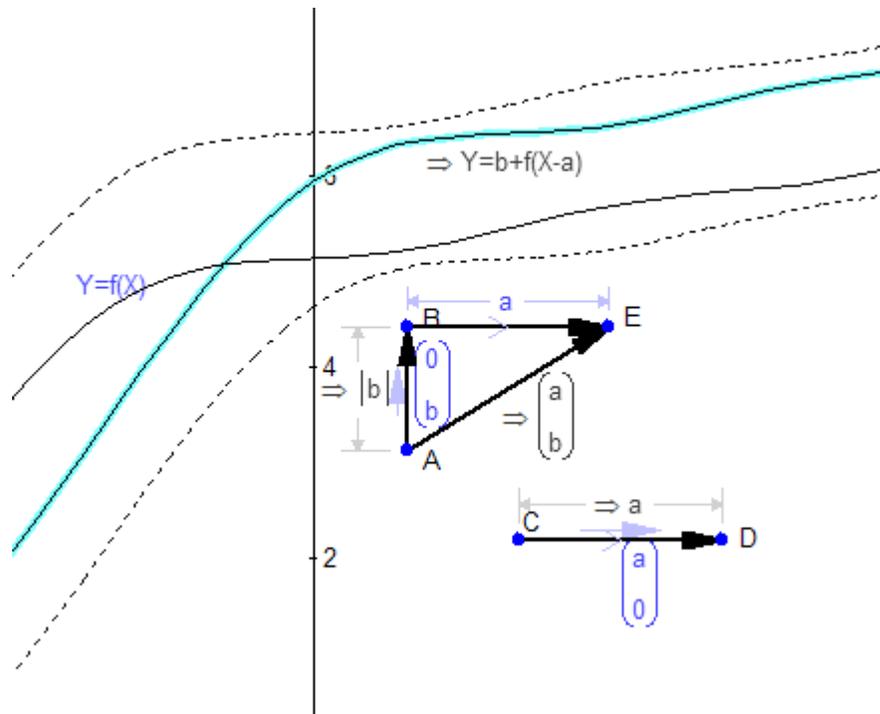
A. It is expected that students will apply prior knowledge from parts 1 and 2, and will be able to write in either order:

$$y = f(x) \xrightarrow{(0,b)} y = f(x) + b \xrightarrow{(a,0)} y = f(x - a) + b, \text{ or}$$

$y = f(x) \xrightarrow{(a,0)} y = f(x - a) \xrightarrow{(0,b)} y = f(x - a) + b$. Students then can check their findings using the software – click the curve and select **Implicit Equation** from the **Calculate** toolbox.

Q3. Is it possible to derive a graph of the function $y = f(x - a) + b$ from the graph of the function $y = f(x)$ in one step?

A. It is expected that students will conjecture that a translation by a vector (a, b) will transform one graph into another in one step. Students can check that using the software. In order to do that students will need to add two vectors of translation and then translate the original graph of function by the resultant vector to verify that it will coincide with the function $y = f(x - a) + b$



Part 4 – Applications and Assessment Problems

Q1. Given two functions, $f(x) = \sqrt{x-a}$ and $g(x) = \sqrt{x+b}$.

- Can graph of one of the functions be derived from the graph of the other function by translations?
- If yes, by how many ways?
- If yes, write the vector of translation for the graph of f to be transformed to the graph of g and visa versa.
- Confirm your results with the help of the software

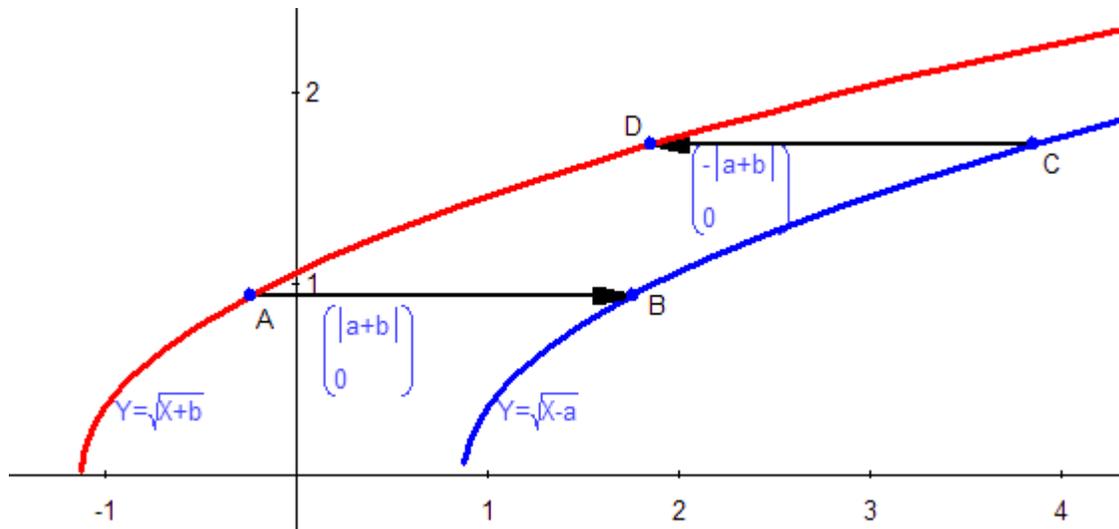
A.

- The graph of one of the functions can be derived from the other by translation along the x -axis
- There are two translations, one from graph of f to g , and another one from g to f , if $a + b \neq 0$. If $a + b = 0$ and thus $f(x) = g(x)$ then there is only one translation.
- The vectors of translation are $(\pm|a + b|, 0)$
- Here are the steps of construction:
 - Click **Function** from the **Draw** toolbox and type $\text{sqrt}(X-a)$ for the function $f(x)$. Click OK. Select the graph of the function, right-click and select **Properties** from the context menu, select Line Style – Solid 2, Line Color – Blue.
 - Click **Function** from the **Draw** toolbox and type $\text{sqrt}(X+b)$ for the function $g(x)$. Click OK. Select graph of the function, right-click and select **Properties** from the context

menu and select Line Style – Solid 2, Line Color – Red.

- Click **Vector** from the **Draw** toolbox and construct a vector AB anywhere on the positive y coordinate plane. (The translation vector needs to reside in the valid domain of the function; otherwise *Geometry Expressions* will map the vector's endpoint to an indeterminate location.) Click the select arrow. Select the vector, choose **Coefficients** from the **Constrain** toolbox and enter coefficients $|a + b|, 0$ for the vector.
- Select point A and the graph of $g(x) = \sqrt{x+b}$. Select **Incident** from the **Constrain** toolbox. Point A will lie on the graph of the function. Point B will automatically be placed on the graph of $f(x) = \sqrt{x-a}$. Students can drag vector AB along the curve to see that the endpoints of the vector lie on both curves for all points on both graphs.

Likewise, construct the opposite vector, $(-|a+b|, 0)$ and attach it to function $f(x)$.



Q2. Given two functions, $f(x) = \ln(ax)$ and $g(x) = \ln(bx)$, $a > 0, x > 0, b > 0, x > 0$

- Can the graph of one of the functions be derived from the graph of the other function by translations?
- If yes, by how many ways?
- If yes, write the vector of translation for the graph of f to be transformed to the graph of g and visa versa.
- Confirm your results with the help of the software

A.

- The graph of one function can be derived from the other one by translation along the y-axis
- There are two translations, one from graph of f to g , and another one from g to f , if $a \neq b$. If $a = b$ and thus $f(x) = g(x)$ then there is only one translation.

- The vectors of translation are $\left(0, \pm \ln \frac{a}{b}\right)$

d. Here are the steps of construction:

1. Click **Function** from the **Draw** toolbox and type $\ln(a \cdot X)$ for the function $f(x)$. Click OK. Select the graph of $f(x)$, right-click and select **Properties** from the context menu, select Line Style – Solid 2, Line Color – Blue.
2. Click **Function** from the **Draw** toolbox and type $\ln(b \cdot X)$ for the function $g(x)$. Click OK. **Note:** If the default values of a and b are set to the same value, (check the **Variables** toolbox), the two functions will be superimposed. Select one of the variables in the **Variables** list and change its value in the edit window below to separate the curves. Select the graph of $g(x)$, right-click and select **Properties** from the context menu, select Line Style – Solid 2, Line Color – Red.
3. Click **Vector** from the **Draw** toolbox and construct a vector AB anywhere on the positive x coordinate plane. (The translation vector needs to reside in the valid domain of the function; otherwise *Geometry Expressions* will map the vector's endpoint to an indeterminate location.) Click the selection arrow. Select the vector, choose

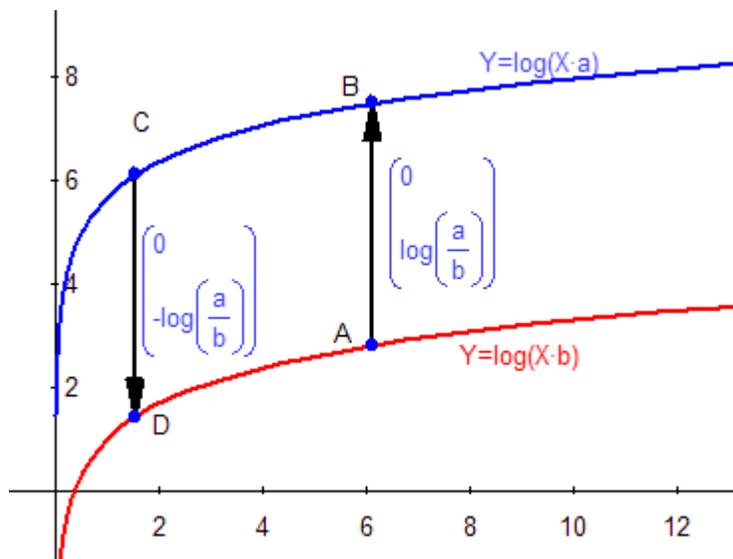
$$\begin{pmatrix} 0 \\ \ln\left(\frac{a}{b}\right) \end{pmatrix}$$

Coefficients from the **Constrain** toolbox and enter for the vector.

4. Select point A and the graph of $g(x)$. Select **Incident** from the **Constrain** toolbox. Point A will lie on the graph of $g(x)$. Point B will automatically be placed on the graph of $f(x)$. Students can drag vector AB along the curve to see that endpoints of the vector lie on both curves for all points on both graphs.

$$\begin{pmatrix} 0 \\ -\ln\left(\frac{a}{b}\right) \end{pmatrix}$$

Likewise, construct the opposite vector, and attach it to function $f(x)$.



Q3. Given two functions, $f(x) = ax + b$ and $g(x) = cx + d$, $a \neq 0, c \neq 0$

- a. Can the graph of one of the functions be derived from the graph of the other function by

translations?

- b. If yes, by how many ways?
- c. Can you specify a translation vector for the graph of f to be transformed to the graph of g and visa versa?
- d. Confirm your results with the help of the software

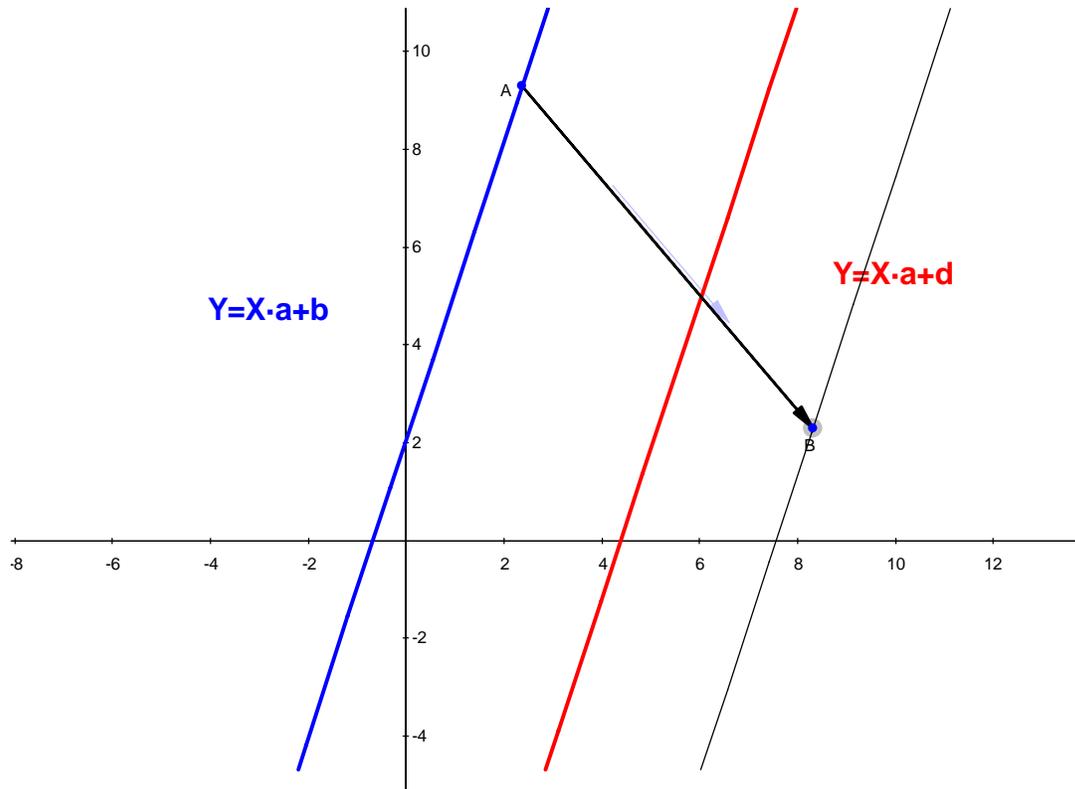
A.

- a. Yes if $a = c$.
- b. There are an infinite number of vectors that connect any two points on the two lines.
- c. Students can specify any vector with endpoints on these two lines or give the general form of the vector between these two lines
- d. Here are the steps of construction:

1. Click **Function** from the **Draw** toolbox and type $a \cdot X + b$ for the function $f(x)$. Click OK. Select the graph of the function, right-click and select Properties from the context menu, select Line Style – Solid 2, Line Color – Blue.

Comment: if the student already determined that lines should be parallel, he/she should type the line with the same slope. If not, the student will start with the general equations of the lines and will arrive at the conclusion that their slopes are the same and edit the equation of the function to fit this conclusion.

2. Click **Function** from the **Draw** toolbox and type $a \cdot X + d$ for the function $g(x)$. Click OK. Select the graph of the function, right-click and select Properties from the context menu, select Line Style – Solid 2, Line Color – Red.
3. Click **Vector** in the **Draw** toolbox and construct a vector AB anywhere on the plane. Click the selection arrow.
4. Select point A and the graph of $f(x)$. Select **Incident** from the **Constrain** toolbox. The point A will appear on the graph of $f(x)$.
5. Select graph of $f(x)$, choose **Translation** from the **Construct** toolbox and click vector AB. The translated image of graph of $f(x)$ will appear on the screen.



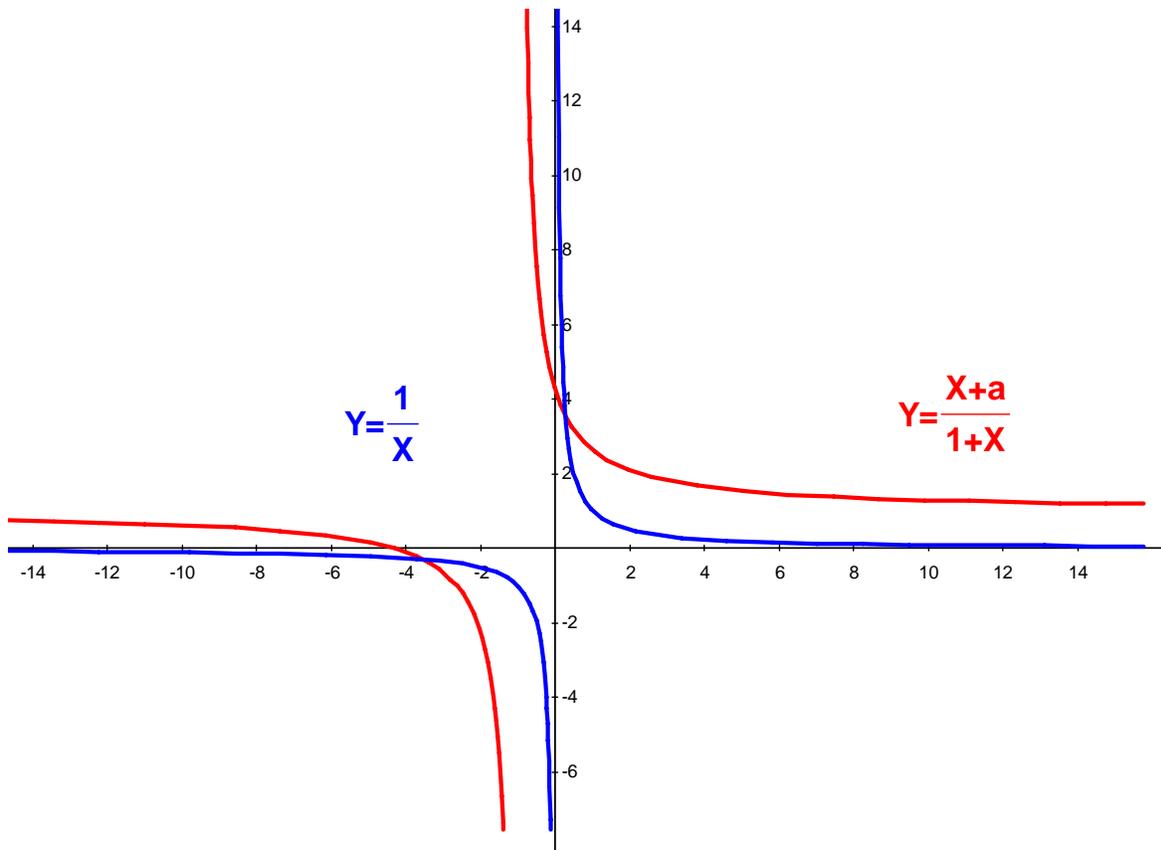
6. Move point B around to overlap the translation with the graph of function g to see various ways that the graph of f can be translated to graph of g .

Extension:

Q1. Given two functions, $f(x) = \frac{1}{x}$ and $g(x) = \frac{x+a}{x+1}$. What is value of parameter a and the translation vector needed to equate these two functions. Investigate this problem with the help of the software

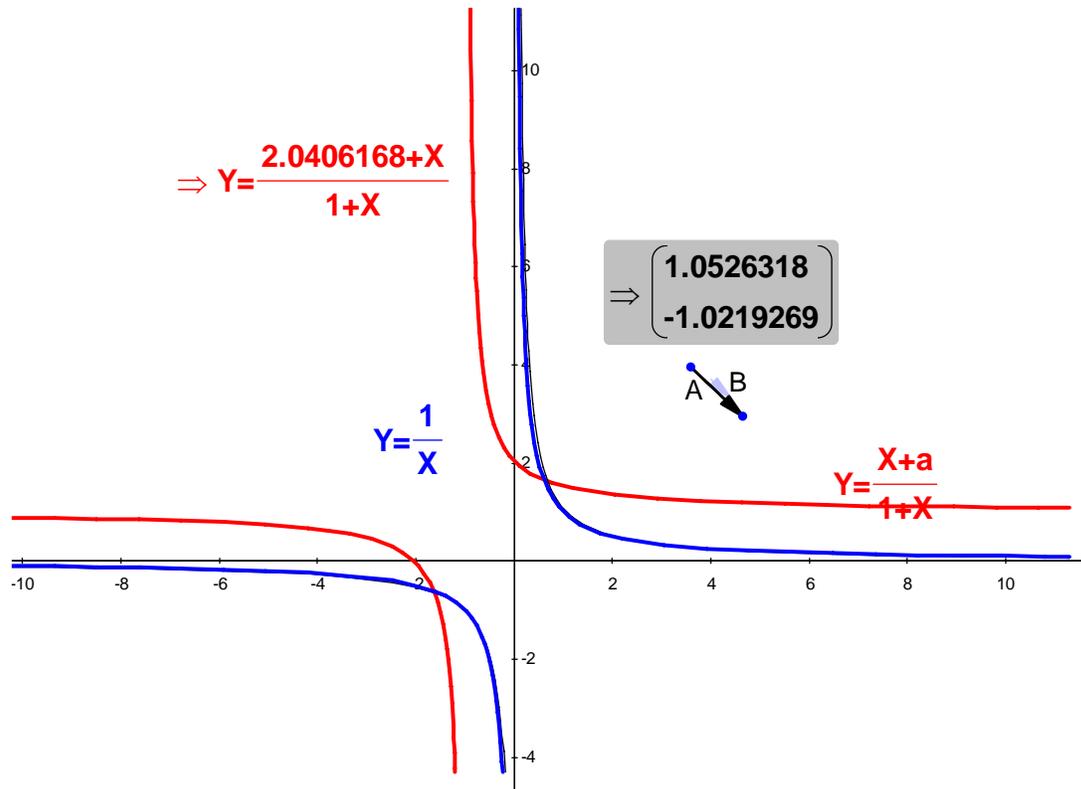
A. Here are the steps of construction:

1. Click **Function** from the **Draw** toolbox and type $1/X$ for the function $f(x)$. Click **OK**. Select the graph of $f(x)$, right-click and select **Properties** from the context menu, select Line Style – Solid 2, Line Color – Blue.
2. Click **Function** from the **Draw** toolbox and type $(X+a)/(X+1)$ for the function $g(x)$. Click **OK**. Select the graph of $g(x)$, right-click and select **Properties** from the context menu, select Line Style – Solid 2, Line Color – Red.



Note: The students' graphs may look different to the image above. Click a in the **Variables** list, then manipulate the slider to change the shape of $g(x)$.

3. Click **Vector** in the **Draw** toolbox and construct a vector AB anywhere on the plane. Click the selection arrow.
4. Select the graph of function $g(x)$ (red), click **Translation** in the **Construct** toolbox, then click the vector AB . The image of the graph of $g(x)$ will appear on the screen.
5. Vary parameter a and vector AB until the translated image coincides with the graph of the function $f(x)$ (blue). In order to vary parameter a and change the shape of the function g , click-and-drag the graph. A drag handle, labeled a , appears on the curve. (Alternatively, you can vary the value of a in the **Variables** toolbox as described above.) Vary the direction of the translation by moving points A and B .



6. When the desired position is found, students can observe the value of parameter a in the **Variables** list. They can also calculate the **Real** equation for the function $g(x)$ by selecting the graph of the function, choosing the **Real** tab in the **Calculate** toolbox and selecting **Implicit equation**.
7. In order to calculate the translation vector, students select the vector, choose the **Real** tab in the **Calculate** toolbox and select **Coefficients**. The numerical coefficients of the vector will appear on the screen. All answers shown on the screen are numerical approximations and allow students to make a conjecture about exact solutions.

Students now can make a conjecture about the values of parameter a and the translation vector. They should confirm this through analytical work:

$$g(x) = \frac{x+a}{x+1} = \frac{x+1+a-1}{x+1} = 1 + \frac{a-1}{x+1}. \text{ Translation exists only when } a = 2, \text{ so that}$$

$$\frac{a-1}{x+1} = \frac{1}{x+1}. \text{ Then, } g(x) = 1 + \frac{1}{x+1}. \text{ Thus the graph of } g(x) \text{ can be derived from the graph}$$

of $f(x) = \frac{1}{x}$ by translation by the vector $(-1, 1)$. Thus, the inverse translation that

transforms graph of $g(x)$ into the graph of $f(x)$ is accomplished by the vector $(1, -1)$. In order to verify that, students can substitute these values for the parameter a and vector AB into software.

By analogy with given examples the teacher can point out the associative property of translation. Students can verify that the order of translation does not affect the result. This property can be combined with the commutative property of translation.

