Triangle Areas

Skills:

- Area formulas
- Similar triangles
- Geometry theorem proving

Midpoint Triangle

Draw a triangle, and constrain its side lengths to be a,b,c. Now join the midpoints of the sides to create a smaller triangle.

What is the area of the smaller triangle?

How does this relate to the area of the original triangle?

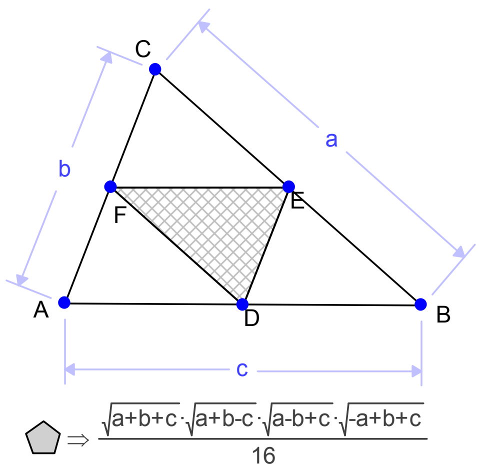


Figure 1: The area of the triangle formed by joining the midpoints of the sides of the original triangle

Can you prove this relationship?

[Hint: look at the triangles ADF, CEF, BDE and think about similar triangles.]

Triangle formed by points a fixed proportion along the sides

Now let's examine the case where instead of D,E,F being half way along their respective sides, they are a third of the way.

Instead of using the midpoint construction to create D, you should place it on the segment and then use the Proportional constraint with a value 1/3.

What is the area of this triangle?

How does it relate to the area of the original triangle?

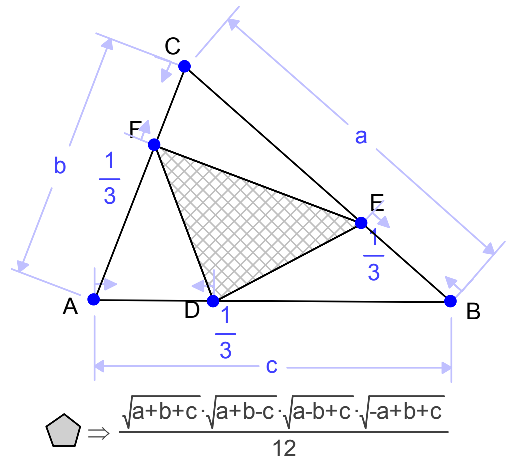


Figure 2: Triangle formed by points 1/3 of the way along the sides

What proportion of the total area is the area of triangle ADC?

What proportion of the area of triangle ADC is the area of triangle ADF?

What proportion of the total area is the area of ADF?

Can you prove the relationship between the area of DEF and the area of ABC?

Generalization

Now what if instead of ½ or 1/3, the vertices of the small triangle are proportion p along the sides?

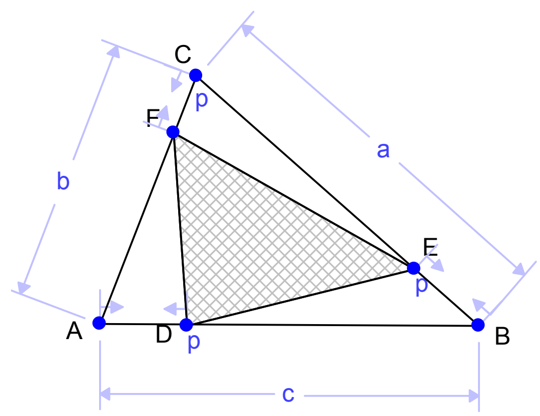


Figure 3: Triangle formed by points proportion p along each side

What is the relationship between the area of the small triangle and that of the original triangle?

Can you verify that this gives the same answer as we got above for the special cases where p=1/2 and p=1/3?

Can you think of any other special cases you could use for verification?

What value of p would put D on top of A? In which case, where would E and F be? And what would the area of DEF be?

What value of p would put D on top of B? In which case, where would E and F be? And what would the area of DEF be?

Can you prove the relationship between the area of DEF and that of ABC?

[Hint: try using the same method as you used for the case where p=1/3]

What value for p would make the area of the smaller triangle half the area of the bigger triangle?

Further Generalization

We can generalize this result even more. Instead of D E and F each being the same proportion along the sides, make each one a different proportion.

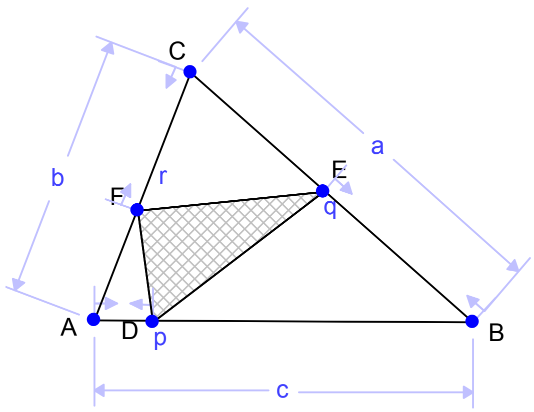


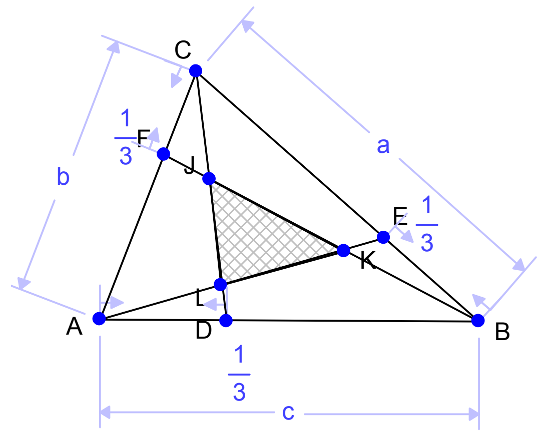
Figure 4: Triangle formed by points proportion p, q and r along the sides

What is the relationship between the area of DEF and the area of ABC?

A Similar Problem

Here is a problem with a similar flavor. Instead of creating the triangle DEF, join each of D,E, and F to the opposite vertex of the original triangle (fig. 5). Now create the triangle JKL from the intersections of these lines.

What is the relationship between the area of JKL and the area of the original triangle?



$$\Rightarrow \frac{\sqrt{a+b+c}\cdot\sqrt{a+b-c}\cdot\sqrt{a-b+c}\cdot\sqrt{-a+b+c}}{28}$$

Figure 5: Triangle formed by intersections of the lines joining the vertices to the trisectors of the opposite sides.

Can you prove geometrically that the area of JKL is the same as the sum of the areas of CFJ, ADL and BEK?

Can you prove the relationship between the area of JKL and that of ABC?

[Hint: this is harder…use similar triangles and don't be afraid to experiment with measuring in Geometry Expressions.]

More Generalization

How could you generalize this result?

Can you find a specific configuration which yields a triangle half the area of the original?