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Chapter 1 – Introduction

New technologies provide exciting opportunities for teachers to enhance student learning. Interactive computer software is one such technology and can be combined with interactive white boards to foster small and large group interaction. Such dynamic interaction provides opportunities for students to access mathematics in new ways and supports a greater range of learning styles. The dynamic interaction provides teachers with an additional component to their repertoire of instructional strategies. Interactivity engages learners and assists in supporting independent student investigations. Using *Geometry Expressions* (GX), for example, students can readily explore concepts, verify numerical solutions to geometry problems and explore properties of geometrical figures. Such activities support students search for patterns and construction of conjectures. These exercises correlate strongly with the process strands of common mathematics standards as well as the underpinnings of algebraic reasoning.

I believe that students should be empowered as much as possible to take control of their own learning. It is through play that conceptual development is often the most rapid and long lasting. Vygotsky observed that children in play displayed intellectual development beyond what they showed in a school environment. (Kozulin 2007) Students engaged in inquiry and discovery activities may well be involved in intellectual play. A feature of the “play” environment is the capacity of the player to manipulate the objects and try things out. It is a safe environment in which to experiment. A strong case can be made for intellectual play, a situation enhanced by the use of appropriate empowering technologies.

Unfortunately many of our high school students have little opportunity to play, particularly in mathematics and particularly where the focus is on the ability to perform for the test. Many students see school as a drudgery to be waited out. Where this occurs it is sad and any way the spark of curiosity and then discovery can be reignited is a blessing. Interactive geometry programs are well known and powerful contributors to student learning. They provide a rich environment for exploration of geometric relationships. GX adds something new to the genre of interactive geometry as the construction process is more intuitive and potentially accessible to many more students.

There are aspects of many interactive geometry programs that are difficult for the novice (student) user. For example creating a construction from scratch often requires a lot of geometrical knowledge which is beyond the student. To allow access to the benefits of Interactive geometry programs students are often presented with pre-prepared sketches and students then play within the limited field these sketches provide. Pre-prepared sketches hide much of the intellectual work involved in creating the drawing in the first place. It seems as if the power of the software is just being tapped rather than unleashed. GX offers a potential pathway to loosen the leash.

The activities in this book are based around GX. To draw upon a culinary metaphor, the book is intended to be a buffet with the teacher picking and choosing a menu that fits the whims and tastes of their class. The activities are wide ranging and touch much of the geometry and trigonometry taught in High School mathematics courses. The ability to solve trigonometry problems in an intuitive drawing / construction mode is exciting as it opens new ways of learning and teaching. When students are able to discover for themselves the ideas and relationships of geometry, learning outcomes particularly in terms of retention and understanding are much improved. Of course, there is no one approach that will work for all students. Some require more hand-holding particularly when working with a new and unfamiliar tool while others need only the briefest of exposures to see how the use of a new tool enables their learning.

With these experiences in mind, the activities in this book involve the use of GX in high school mathematics classes. The materials are designed to be flexible. They may be used as the basis for an independent geometry unit, as a set of labs for reinforcement or enrichment for a text based unit, as individual activities to be woven into the mathematics curriculum on appropriate occasions, as a support for differentiated instruction in the form of self-guided discovery and extension or as the basis for contextualized learning activities.

Content and Organization

Chapters 2 – 4 discuss key features of GX and the pedagogy underpinning the lab activities in chapters 5 to 10. GX is described as an interactive symbolic geometry program. Interactive geometry is a software genre with well known commercial packages, shareware programs and open source offerings. The features of this genre are documented elsewhere. These introductory chapters focus on *Constraints* (Chapter 2) and *Symbolics* (Chapter 3), features that extend interactive geometry and a scenario (Chapter 4). The scenario is a passionate appeal for the use of an investigative approach with the teacher becoming more of a guide on the side.

Chapters 5 – 10 are the lab activities. They are organized in themes which are likely to be covered in a number of high school mathematics courses. These themes are content based and develop from Middle school or beginning high school through to topics in advanced placement courses. For example the chapter on trigonometry begins with the Pythagorean Theorem and extends through right triangle trigonometry, the sine formula, law of cosines and polar coordinates. The chapter on locus begins with the circle, and ends with Bezier curves. Each of these chapters is prefaced with an outline of the labs. Each lab has teacher notes to assist in planning.

The themes and the chapters are:

- Chapter 5 Congruence and Similarity
- Chapter 6 Proof
- Chapter 7 Transformations
- Chapter 8 Trigonometry
- Chapter 9 Coordinate Geometry
- Chapter 10 Loci

While the labs are self-contained, the development within each chapter assumes a growing knowledge of GX and ability to construct the drawings. Consequently initial labs have quite detailed instructions while later labs have less detail, assuming that students have prior experience with the drawing tools required. This is consistent with encouraging students to become increasingly independent and to use the software to support their own learning. Consequently some preliminary instruction may be required for advanced classes beginning with later activities within the chapters. Using interactive white boards has been found to be very effective in demonstrating the drawing tools in GX.

The student manual (in press) provides a subset of the activities in this book associated with a particular course of study.

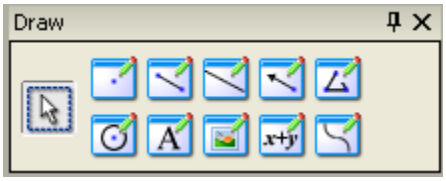
The activities in each unit provide a framework for students' explorations. Often there are one or two activities requiring very little technical expertise that you will feel comfortable giving your students with little or no guidance beforehand. Other activities maximize guided discovery by students. Some of these are more open-ended than others, and their difficulty levels vary. You should choose those activities that are most appropriate for the ability and skill levels of your students. The text provides blackline masters for student work.

Terminology:

GX can be used with menu options or icons. The icons arranged in toolboxes duplicate most of the menu options. The icons are shown sometimes, often the first time a tool is used in the text and the menu item or description used thereafter. The menu item is referred to using the vertical bar to separate levels. For example
Edit | Settings | Math | Math | Angle Mode means

- choose the Edit menu,
- choose Settings from the Edit menu,
- choose the Math tab in the left sidebar
- open ⊞ → ⊟ the Math window
- choose Angle Mode from the Math window.

Draw | Line Segment is the icon from the Draw toolbox.



The first time a particular construction or program feature is used a detailed description is given. A statement of the construction is made and then the steps detailed in point form. Where the feature is used subsequently the point form detail is not provided.

For example in an early Lab:

Construct a line segment of length 4.

From the Draw menu select the Line Segment tool .
Click, drag and release is one way to draw the line segment.

Choose the Select tool .
Click on the line segment.

Click the Constrain Distance/Length tool
Type in 4 and press Enter.

Subsequently the Activity might just say
Construct a side of length 4.

Hints and suggestions

If you're new to GX, work the activity yourself in your preparation for working with students. Becoming familiar with the activity can greatly facilitate your own learning of the software and increase your comfort level with these materials in the classroom. *GX*'s online help is useful for clarifying the use of a menu or tool and in understanding GX specific language.

Even if you're using GX for the first time, don't let your inexperience keep you from placing the software into the hands of your students! One of the best aspects of using interactive geometry software is learning along with the students. Invariably when I have had a problem in class it is one of the students who is able to find a solution, teaching me and the rest of the class. Such experiences are valuable in modeling approaches to learning, moving the dynamics of the classroom to a more collaborative learning environment and valuing the skills students bring to the classroom. Nothing in

the classroom can compare to the excitement generated when students and teacher experience the joy of learning mathematics together.

Where an activity has an extension component, there is an opportunity for more open investigation. For the able students this is where the most valuable learning can be gained.

Regardless of the degree of structure you are comfortable with, be sure to allow time for students to simply explore with the software. Before or after initial activities students enjoy trying the tools and experimenting on their own. This free exploration motivates students to want to learn more about the capabilities of the software and invariably leads to mathematical discovery or consolidation. Where a student holds a geometrical misconception this experimentation can provide the crucial discord required for the student to re-conceptualize and thus move forward with their learning in mathematics.

I encourage you to allow your students to work in pairs or in small groups for some if not most of the activities. The units in this book, as well as GX itself, are ideal instruments for cooperative learning. When we teach this way, our students exhibit both increased mathematical understanding and greater proficiency in using the software.

Chapter 2 – Constraints

Constraints is a feature distinguishing GX from other dynamic geometry programs. This chapter briefly describes attributes in drawings that can be constrained, an application using *Constraints* as a pedagogical approach compared to other interactive geometry programs, the use of *Constraints* to solve a problem and to develop understanding of a problem. These examples indicate reasons why *Constraints* have application in high school mathematics courses.

Think of *Constraints* as a measurement that we fix in our drawing. For example, when drawing or constructing a figure on paper using drawing instruments we draw lines of particular lengths (using ruler and compass) and create angles of particular measure (using a protractor or template). Constraints mirror this process in an electronic construction. We can draw a line and then set its length or select intersecting lines and set the angle. There are many more attributes that can be constrained,

In addition GX allows the user to constrain attributes such as coordinates of points, perpendicularity, parallel, direction or slope of lines, vectors, circles, coincidence, tangency, congruence, equation and point proportional. The toolbox to the right illustrates the range of properties that can be constrained.

Using *Constraints* is closely aligned with traditional construction techniques. Consequently I find students are more able to create drawings independently of the teacher than with other dynamic geometry software. The following example involves exploring the converse to the Pythagorean Theorem and compares the use of a non-constraint based interactive geometry system with the use of *Constraints*.

Explore the converse of the Pythagorean Theorem:

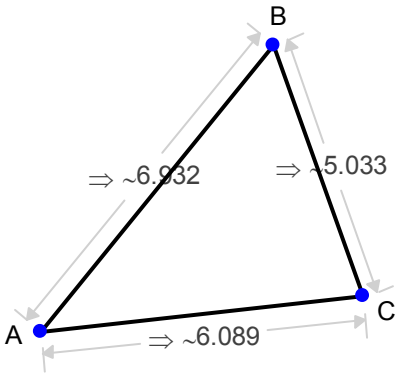
How do we know whether or not a triangle has a right angle given the lengths of each side? A teacher might begin by posing the question, “Is a triangle of sides 5, 6 and 7 right angled?” Clearly drawing an accurate sketch will give the student a feeling for whether or not this is the case with a follow up using the theorem.

To draw a sketch using dynamic geometry software I would ask the students to

- draw a triangle,
- measure the lengths
- drag vertices until they get a triangle close to the desired measurements.

The diagram on the right shows such a sketch.

The sketch can be used to explore other side lengths by dragging the vertices to change the side lengths.



Note that the side lengths will not fit the original problem specifications exactly but it is close enough to be convincing that there is no right angle in a 5 – 6 – 7 triangle. Dynamic Geometry software enables this exploration to be done readily and intuitively by students.

LAB # 1 *Three sides*

Aim: Investigate whether or not knowing the lengths of three sides is sufficient to ensure that the factory will construct the window to fit.

ALPINE GLASS
Crystal Clear Custom Windows

Alpine Glass specializes in those difficult to make windows.

Aspen Lodge has asked for their premium double glazed panels for their unique architect designed ski lodge. The triangular windows need to be measured up by Cherie, Glacier Glass' local representative. The measurements will be sent back to the factory for the custom double glazed panels to be manufactured.

Cherie, Glacier Glass' local representative, measures the three sides. She measures 4, 6 and 8 feet.

Part 1

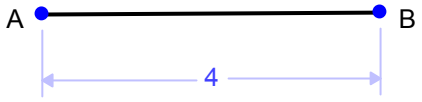
Open *Geometry Expressions* and start a new sketch File | Open.

Construct a side of length 4.

From the Tools panel select the Draw | Line Segment tool .
Click, drag and release is one way to draw the line segment.

Choose the select tool .
Click on the line segment.

Click the Constrain | Distance/Length tool
Type in 4 and press Enter.

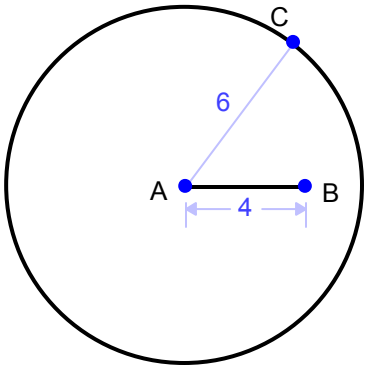


The line segment has endpoints labeled A and B and is of length 4 units.

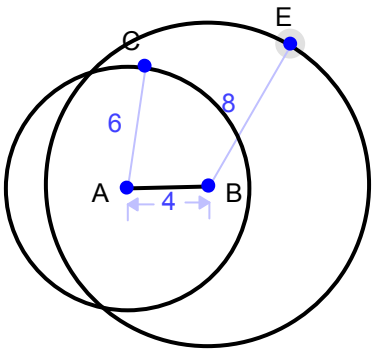
Draw a circle with center A and radius 6.

From the Draw toolbox select the Circle tool .
Click on A, drag and release.
Select tool
Click on A and then Ctrl-click on C

Click the Constrain | Distance/Length tool
Type in 6.



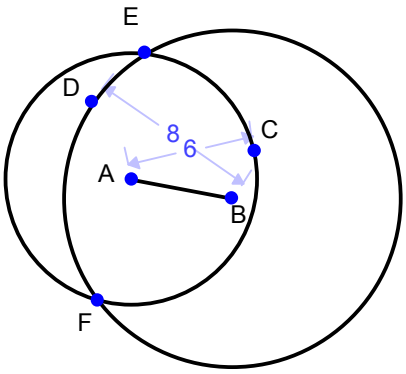
Draw a circle center B and radius 8.



Mark points 6 units from A and 8 units from B.

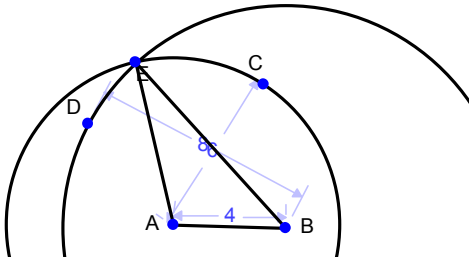
These are the points of intersection of the circles.

Select both circles by clicking on one and then holding the shift key down click on the other. From the Construct menu select the intersection tool .



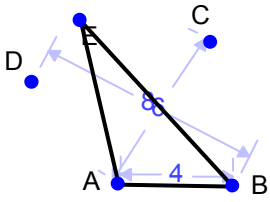
It is also possible to choose the point tool and when the mouse is in the right spot both circles are selected and clicking will mark the intersection.

Draw line segments AE and BE



Hide the construction circles

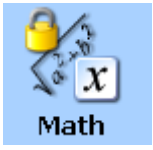
Select tool
Click on circle
Right click and choose Hide
or View | Hide

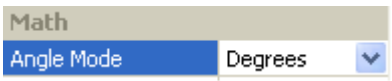


Check that the angle mode is set to degrees, the default setting is radians.

Set the angle mode to degrees

Choose Edit | Settings | Math.
Click the down arrow by Angle Mode to select degrees.



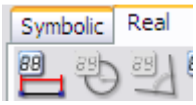


Measure the lengths and angles

Select a side
Choose Calculate (Output) |Real | Length

AB = 4

AE = ____



Repeat for each side.
Select two sides (Ctrl-click)

Choose Calculate (Output) | Real | Angle
Repeat to measure the angles

BE = ____

∠A = _____°

∠B = _____°

∠E = _____°

Drag the vertices of your triangle. Describe what happens. (Think about what changes and what stays the same.)

_____

_____

Part 2

Open a new sketch

Construct a triangle.
Constrain the lengths to 4, 6 and 8.
Measure the angles.

∠A = _____°

∠B = _____°

∠C = _____°

How do the angles compare to your triangle in Part 1? _____

Congruent figures have the same size and shape.

Are the triangles in each sketch congruent? _____

Part 3

Build a triangle with sides 5, 12 and 13 units.
Change the lengths on your current sketch.
(Double click on the measurement, and then enter the new value).

What kind of triangle is this? _____

Part 4

What happens when you try with sides 4, 5 and 10? _____

Why? _____

_____

(Note: Press Ctrl-Z to undo the previous step)

If Cherie sends the lengths of the three sides will the manufactured window fit?

_____

LAB #1 Three sides

In this activity students:

- Simulate a scale drawing using ruler and compass construction;
- Establish that specifying the three sides ensures the correct window is manufactured (that all constructed triangles are congruent);
- Construct the triangle using *Constraints* and
- Investigate the triangle inequality.

You may like to introduce the context of Glacier glass through a class discussion before beginning the lab activity.

> I.e. Cherie, Glacier Glass' local representative, measures the three sides of the triangular window as 4, 6 and 8 feet. Assuming that Cherie's measurements are accurate and the factory creates a triangular window panel with these measurements accurately, must the window fit? It may be useful to note that in practice the glass would be cut smaller allowing a tolerance for fitting. This would normally be about three eights of an inch.

The instructions for the GX construction are quite detailed so it can also be used as an introduction to drawing in GX. This includes setting the angle mode to degrees as *GX*'s default mode is radians. For a class that has significant experience of GX you may like to develop a condensed activity sheet with outline instructions only.

A simpler way of creating the 4-6-8 triangle would be to draw the triangle and then constrain the sides. However, students would not have the opportunity to explore why knowing the three sides means that only one triangle can be created.

A further discussion point needing to be addressed is mirror images. It is likely that some students will create mirror images and see these as different triangles as it is not possible to superimpose one upon the other. Flipping the window over will allow it to fit. Having two 4", 6" and 8" cardboard triangles already prepared would be helpful. Draw around the model on the board. Ask a student to measure the "window" and record the measurements. Simulate the mirror image being unloaded on site with the other model and then ask for it to be installed.

Solution

Part 1

AE = 6
BE = 8
$\angle A = 104.5°$
$\angle B = 46.6°$
$\angle E = 29°$
The size and shape stay the same, the orientation and position can change.

Part 2

$\angle A = 104.5°$
$\angle B = 46.6°$
$\angle C = 29°$
The angles are the same.
The triangles are congruent.

Part 3

Right triangle

Part 4

It disappears into a line of length 10.

This is because you can not have a triangle where one side is longer than the sum of the other two sides. This is called the triangle inequality.

If Cherie sends the lengths of the three sides the manufactured window **will** fit?