Geometric Measure of Aberration in Parabolic Caustics

Alex Turzillo
Senior, Tigard High School
Tigard, OR

I. Introduction

The mirror in a flashlight or automobile headlight reflects the light from a tiny source – the incandescent filament or LED – and directs it into a single beam. The mirror is necessary to shape the light beam, for the point source without a reflector would project light equally in all directions. Some mirrors are designed to spread light over a large region, while other mirrors focus light into a nearby point (see Figure 1). In situations where the width of the beam increases with the distance from the mirror, the light becomes less intense – less "focused" – at greater distances. In the opposite case where rays of light approach a common point in space, the beam becomes more focused as it narrows. (In physical terms, concentrating the beams means more photons per cross-sectional area.) A mirror's ability to redirect light (focus light) toward a common point (naturally, the "focus" or "focal point") is especially significant in the field of radiotelemetry, the engineering of wireless communications. Satellite dishes reflect a wide beam of radio waves into a focus where the signal is concentrated on a receiver.

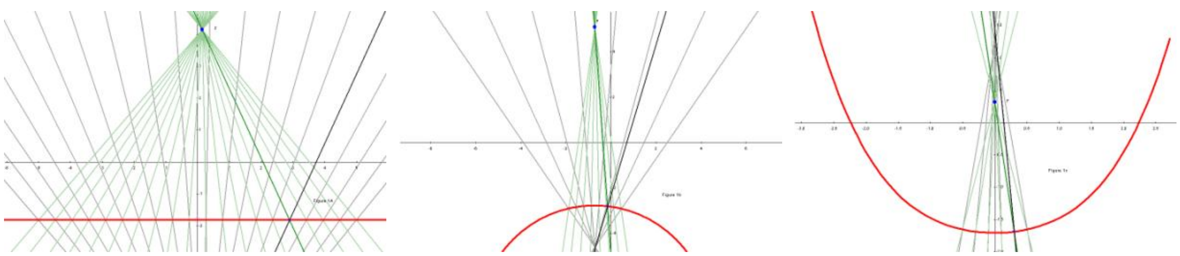


Figure 1

High school physics and geometry classes teach that the parabola – more correctly, the paraboloid of revolution, the surface described by a parabola rotated about its axis of symmetry – is the shape that reflects parallel rays of light into a focus. Conversely, rays originating from the focus reflect and become parallel. (In optics, this is known as "collimation.") In order for either of these phenomena – focusing and collimation – to take place, the collimated light rays must also be parallel to the paraboloid's axis of symmetry. This fact poses questions: what happens when collimated light enters the mirror at an angle, and what happens when light emanates from a point other than the geometrically defined focus? High school level geometry can provide the answers.

Working as an intern at Saltire Software last summer, I had the opportunity to investigate this problem and others with the company's software, Geometry Expressions, which combines a computer algebra system with a geometry engine and allows teachers, engineers, and students like myself, to solve problems in geometry algebraically and also numerically. One can draw a geometric system, constrain quantities (measurements, proportionalities) symbolically or numerically, and request output in the form of lengths, coordinates, and equations. The system can also be used to construct geometric objects such as reflected rays, loci, and envelopes. It was the combination of the software's variable-crunching and construction capabilities that helped me understand the underlying simplicity of the parabola problem. Saltire Software President Phil Todd led me to the article in the March 2008 Mathematics Teacher titled "Teaching Algebra and Geometry Concepts by Modeling Telescope Optics"[1] which posed question, "What if light enters the mirror at an angle?"

The aim of this investigation is to study the caustics (curves formed by reflected light) of the paraboloid by modeling its *xy*-plane curve (a parabola) and to explore the effects that light angle, curve diameter and curve depth have on aberration (inability to focus).

Measuring Focal Ability

Focal ability is the ability of the mirror curve to reflect rays into a small volume called the focal volume. This volume is represented in our model as an area in the same plane as the parabola. We define the focal volume as the set of all points at which two reflected rays may intersect. In the following investigation, we draw conclusions about the focal volume's relationship with variables such as light angle, curve diameter and curve depth.

II. The Parabola

Perpendicular Rays and the Focal Point

We draw a parabola (red) with the equation $y=kx^2$ (for clarity, x- and y- axes are not shown). In Figure 2, we draw a ray (blue) parallel to the y-axis colliding with the parabola at A, and we draw its reflection. As we vary the position of A along the curve, we find that all rays appear to intersect at one point (Z). For example, the ray (green) that intersects the parabola at B reflects (purple) and appears to travel through the same point.

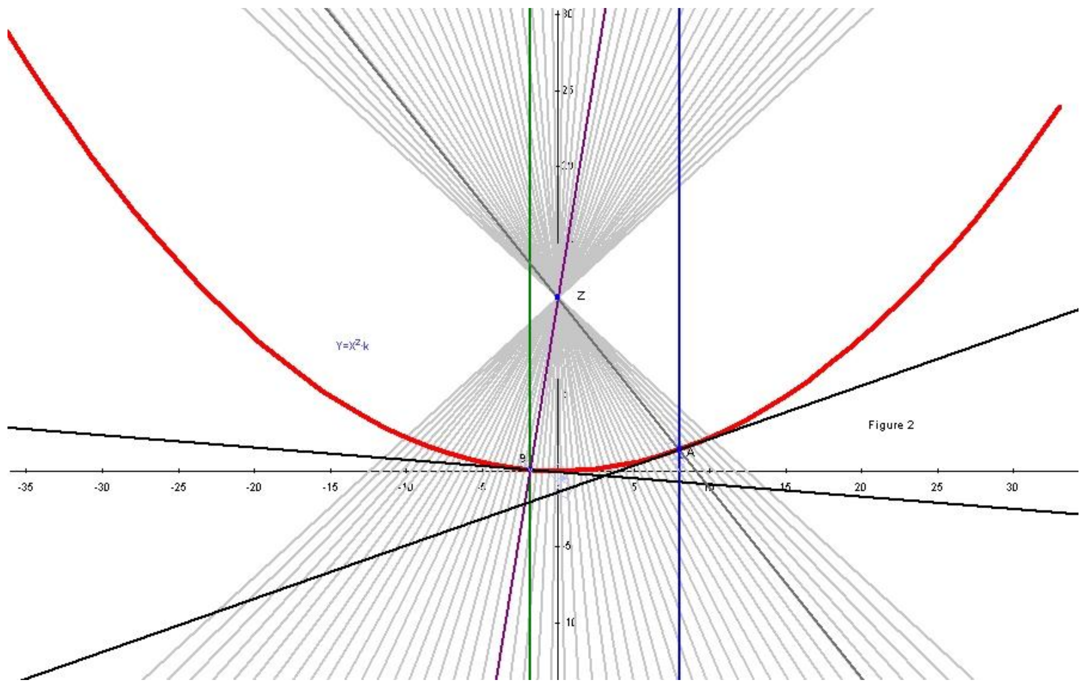


Figure 2

After defining the construction illustrated in Figure 2, we ask the system to solve for the coordinates of the intersection point. Geometry Expressions reveals coordinates that are independent of A and B:

$$f_o = \left[\, 0, \frac{1}{4\,k} \right]$$

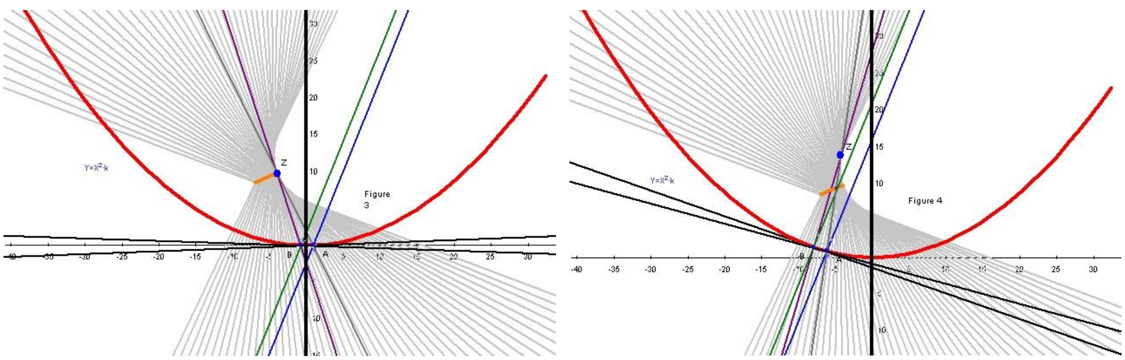
We call this set the optical focus, f_o. From the fact that the optical focus is dependent only on k, a parameter of the curve, we infer that each parabola has one unique optical focus when rays are directed parallel to its axis. We note that the optical focus is equivalent to the mathematical focus in this case, and we will use the relationship later as we explore the properties of focal volumes created by the parabola when light enters at other angles.

Angled Rays

To get an idea of what these focal volumes look like, we draw the set of reflected rays (gray) when incoming rays are directed into the parabola at angles not parallel to its axis of symmetry. We model rays incoming at an angle θ with the directrix perpendicular to the axis of symmetry. The rays do not appear to focus to a point, but rather to an area. The shortest line across the area is called *the bottleneck* (orange).

We define *aberration* as the length of the bottleneck. This definition only approximates the physical situation of the parabolic mirror because it neglects rays that reflect out of the *xy*-plane. Nonetheless, this approximation retains some validity, as we expect rays that reflect in other plane curves of the paraboloid to behave similarly. Reflection components in the *x*- and *y*- directions are invariant across plane curves, and z-components vary to reflect rays toward the plane of the parabola.

Here, as before, we trace the reflections of a pair of parallel incoming rays and observe where those reflections intersect. Unlike in the previous situation, the intersections vary with the pair of incoming rays.

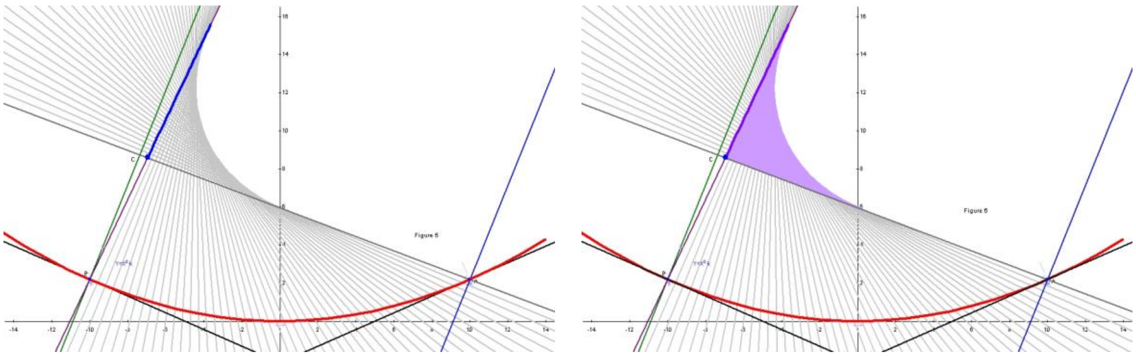


Figures 3 and 4

For a given angle of incoming light (θ), we observe that rays that intersect the parabola near the vertex make reflections that intersect each other near the bottleneck (Figure 3); while rays that intersect the parabola farther from the vertex make reflections that intersect each other farther from the bottleneck (Figure 4). We infer that rays that intersect the parabola on the boundaries of its domain make reflections that intersect each other on the boundaries of the focal volume. We limit the focal volume to a finite size by defining a mirror diameter *d*. Now the *x*-coordinates of reflection points are limited to the set $[-d/2, d/2]$, and the focal volume has definite boundaries and shape.

We here define two significant positions of the intersection of reflected rays, Z: C is the intersection point when A and B lie on opposite ends of the parabolic segment, the very edges of our mirror; D is the limit of the intersection point as A and B approach the vertex of the parabola. In other terms, C and D represent the two ends of the bottleneck.

We determine effects of the incoming ray angle on the shape of the focal volume by creating a graphical representation in Geometry Expressions, sliding two rays over the parabola to sweep out the area of their intersections. To do this, we draw two rays with collision points A and B, both on the left edge of the curve (*x=–d/2*). We construct the locus (blue) of the intersection point (C) as A slides around the parabola to *d/2* (Figure 5). Next, we construct a set of loci (purple area) as B slides to *d/2*, representing the two-dimensional cross-section of the focal volume in the plane of the parabola. The area appears bounded by two lines and a curve (Figure 6).



Figures 5 and 6

For the purposes of this investigation, there is no need to analyze the nature of the bounding curve. We develop a model of aberration based solely on the lines that bound the focal volume. Using the curve parameters *k* and *d* as constraints, Geometry Expressions finds the angle between the bounding lines (α) at C:

$$\alpha = \arctan\left(\frac{-4\,k\,d + 4\,k^3\,d^3}{1 - 6\,k^2\,d^2 + k^4\,d^4}\right)$$

The expression is independent of the angle of incoming rays (θ). This fact will become useful in the following lemmas.

We note that the tangent lines at points A and B on the parabola intersect at a point E. The line segments AB, BE and EA form a triangle. In order to geometrically define the position of C relative to *θ*, we prove first that the locus of C (with respect to *θ*) is a circle and second that it contains A and B as well as the circumcenter of triangle ABE.

Lemma 1. Given ΔABE with ∠AEB defined as *φ*; parallel lines GI, with A, and HJ with B; and ∠BAI defined as *θ*; let C be the intersection of the reflections of GA in AE and HB in BE. ∠ACB≅2*π*-2*φ*, as shown in Figure 7.

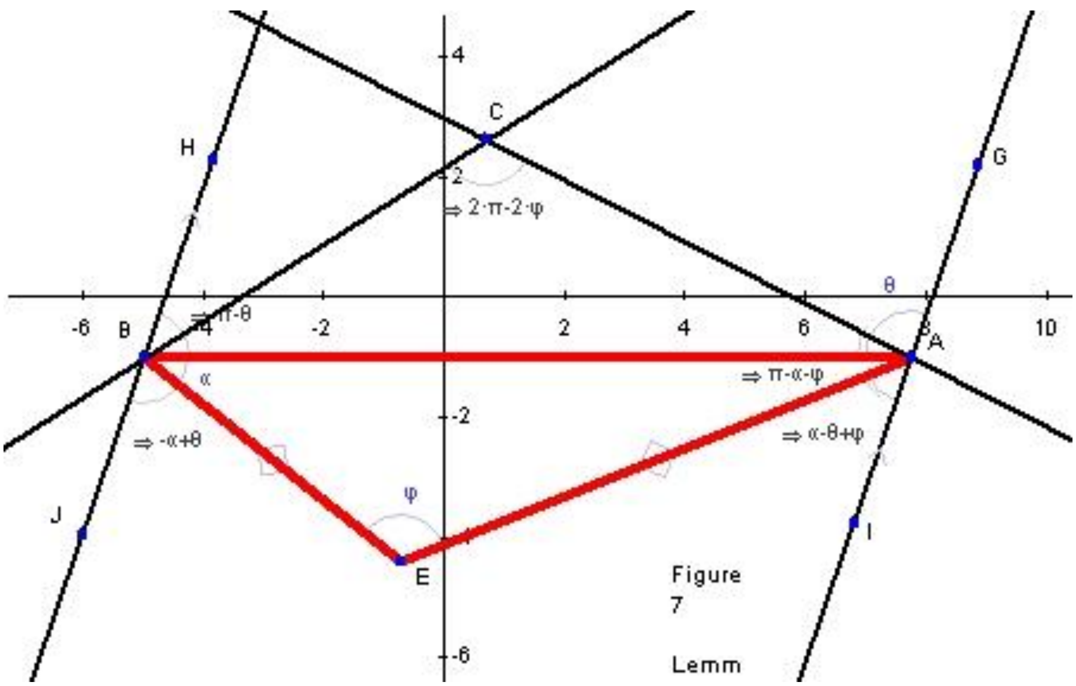


Figure 7: Lemma 1

Proof of Lemma 1. Let ∠ABE be *α*. ∠BAE≅*π*-*φ*-*α* by definition of triangle, and ∠ABH≅*π*-*θ* by corresponding angles. By supplementary angles, ∠EAI≅*α*+*φ*-*θ* and ∠EBJ≅*θ*-*α*; and by reflection, ∠CAE≅∠EAI and ∠CBE≅∠EBJ. ∠ACB≅2*π*-2*φ*, as the sum of internal angles measures in a quadrilateral is 2*π*.

Lemma 2. Given ΔABE with circumcenter K and ∠AEB defined as φ, ∠AKB≅2π-2φ, as shown in Figure 8.

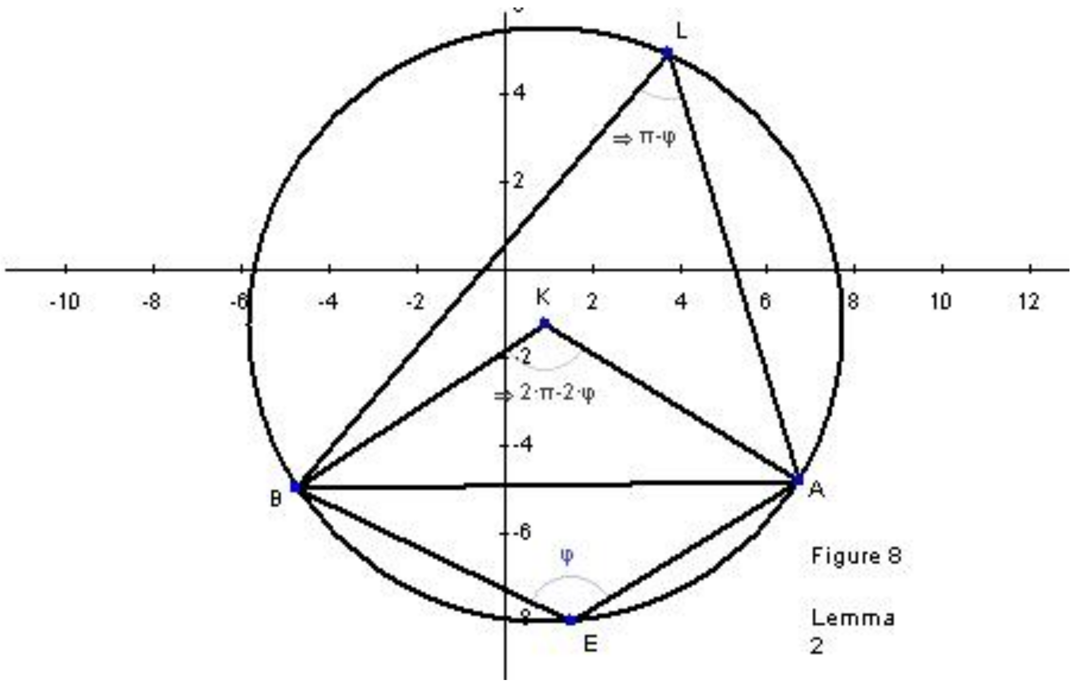


Figure 8: Lemma 2

Proof of Lemma 2. Let L be a point on the circumcircle. ∠ALB≅π-φ by the geometry of a cyclic quadrilateral. ∠AKB≅2π-2φ, as it is the central angle corresponding to the inscribed angle ∠ALB.

Theorem 3. Given ΔABE with circumcenter K and parallel lines GI, with A, and HJ with B; the intersection of the reflection of GA in AE and HB in BE, C, lies on the circumcircle of ΔABK.

Proof of Theorem 3. Proof follows directly from *Lemma 1* and *Lemma 2*.

By construction of the catacaustic, D is the limit point of C as the diameter of the reflector approaches *0*; thus D lies on the circumcircle of some triangle ΔA'B'K' at the limit. Using this model as input, the symbolic geometry system reveals the radii of the two circles, the loci of C and D:

$$\lim_{t \to \left(\frac{1}{2}\right)} \frac{3\,(1+16\,N^2-4\,t+4\,t^2)^2}{8\,(-4\,t^2+4\,t-1+16\,N^2)} = 6\,N^2$$

$$\lim_{t \to 0} \frac{3\,(1+16\,N^2-4\,t+4\,t^2)^2}{8\,(-1+16\,N^2+4\,t-4\,t^2)} = \frac{3\,(1+16\,N^2)^2}{8\,(-1+16\,N^2)}$$

The circles are tangent: they share a point, K, and they are symmetrical about the axis of the parabola by construction. One can see that the coordinates of K are independent of those of A and B by the reflective properties of the parabola[2].

We have so far described the aberration as a distance between point on tangent circles with defined radii. Theorem 6 describes the position of the points on the circles.

Lemma 4. Given triangle ABC with ∠ABC = β and ∠ACB = γ, let L be the circumcenter of ABC and M be the intersection of the circumcircle and perpendicular bisector of AB; ∠CLM is $\pi-\gamma-2\beta$.

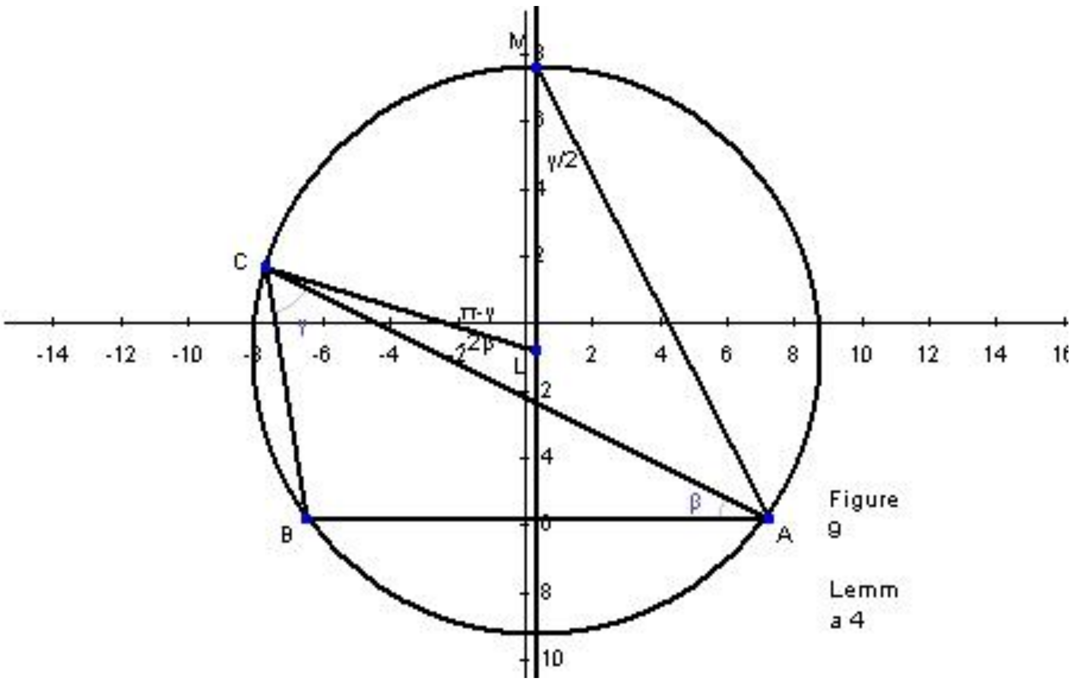


Figure 9: Lemma 4

Proof of Lemma 4. In Figure 9, ∠AML is $\gamma/2$, as, by the geometry of inscribed angles and chords, it is half ∠AMB, which is γ. ∠BAM is $\pi/2-\gamma/2$ by definition of triangle, so, by subtraction, ∠CAM is $\pi/2-\gamma/2-\beta$. ∠CLM is the central angle corresponding to ∠CAM, so it is $\pi-\gamma-2\beta$.

Lemma 5. Given a ray YA reflected at an angle of θ in AE of ΔABE where $\angle$AEB$\cong\varphi$; $\angle Y'AB\cong\varphi-\theta$.

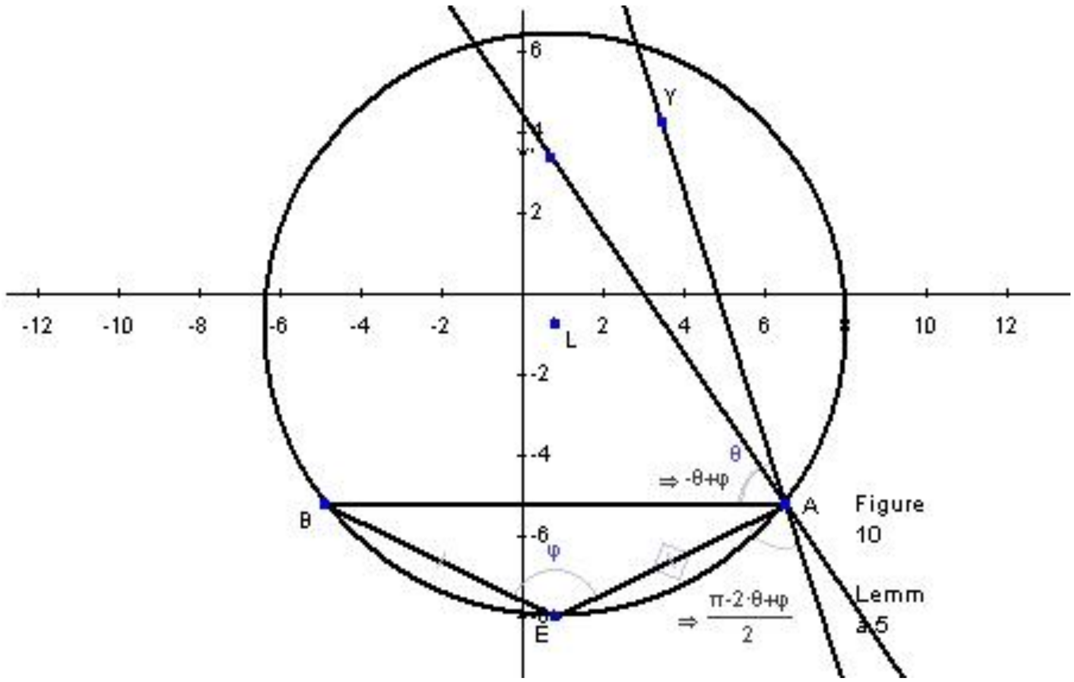


Figure 10: Lemma 5

Proof of Lemma 5. In Figure 10, the angle between the incident ray and AE is $\pi/2+\varphi/2-\theta$ by supplementary angles. The angle between the reflected ray and AE is congruent by reflection, and the angle between the reflected ray and AB is $\varphi-\theta$.

Theorem 6. C and D are collinear with the mathematical focus.

Proof of Theorem 6. Given $\angle$CIJ$\cong\pi-\gamma-2\beta$ from Lemma 4, $\beta\cong\varphi-\theta$ from Lemma 5, and $\gamma\cong2\pi-2\varphi$ from Lemma 2; $\angle$CIJ$\cong2\theta-\pi$, dependent only on θ.

C and D are at equivalent positions along their respective loci; thus the extrapolation of CD includes the circles' point of tangency, the mathematical focus. Having defined the radii and relative positions of the locus circles as well as the relative positions of C and D on the circles, we now have enough information to model the aberration as a length CD.

Converting our geometric input and algebraic constraints into an output, the symbolic geometry system yields

$$L = \left| \frac{1}{4} \frac{\cos(\theta)\, k\, d^2 \,(d^2 k^2 + 3)}{d^2 k^2 - 1} \right|$$

Because *d* and *k* are parameters of the mirror and are independent of the incoming ray angle, *L* is directly proportional to the absolute value of the cosine of that angle. As the angles falls away from *π/2*, the magnitude of aberration increases as rays focus into a volume rather than a point.

Figure 11 contains a graph of the aberration function.

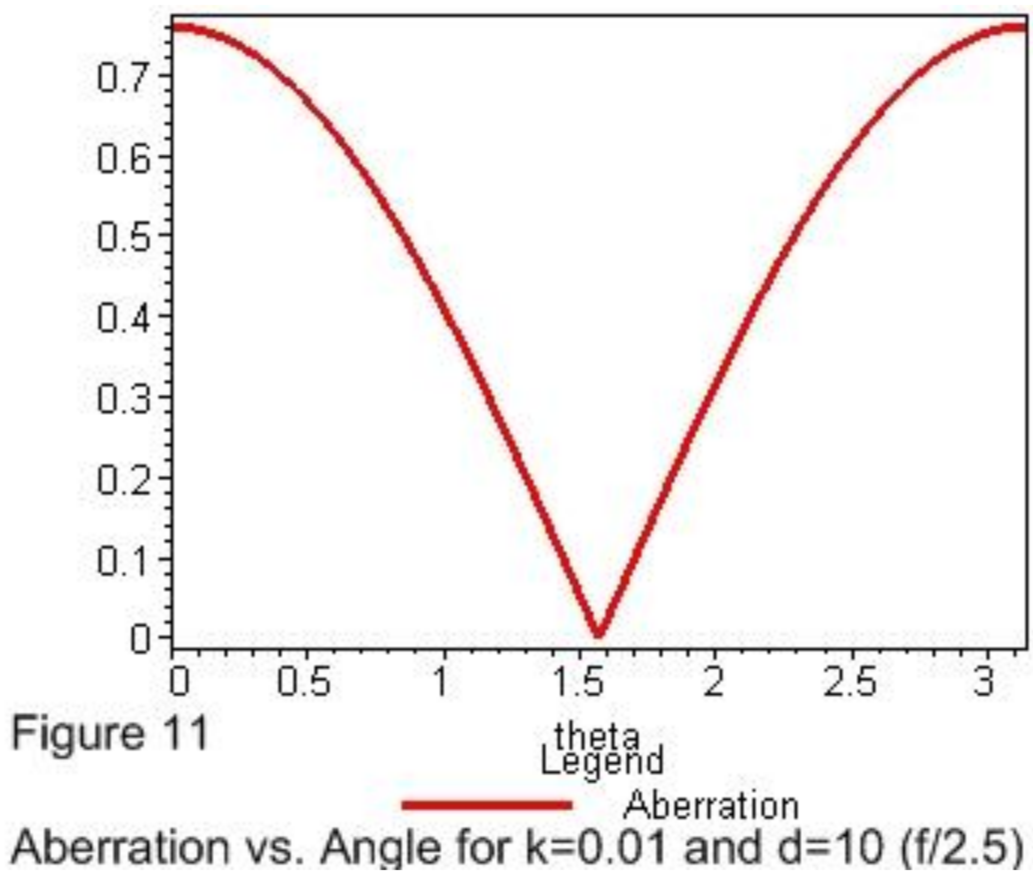
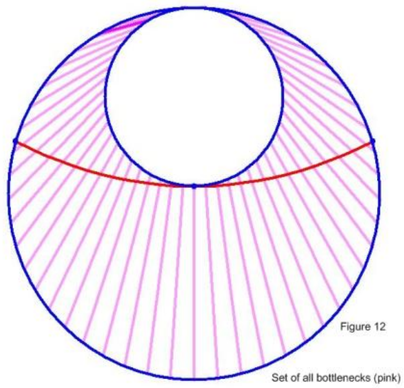


Figure 11

Models of aberration at varying angles are shown by the pink lines in Figure 12 and the difference of chords in Figure 13.



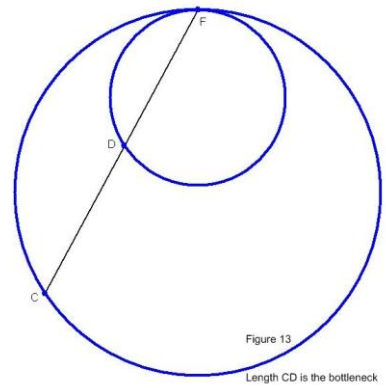


Figure 12: Set of all bottlenecks (pink)

Figure 13: Length CD is the bottleneck

III. Conclusion

Symbolic geometry software allowed us to define *aberration* in terms of elementary geometry. Specifically, the technological tool aided the simplification of the scenario in which light rays reflect from a parabolic surface to a basic representation consisting of circles, triangles and lines. The ability of Geometry Expressions to make connections between the two models allowed us to delve deeper into the mathematics of parabolic mirrors without resorting to excessive computer algebra.

References

[1]Siegel, Dickinson, Hooper and Daniels. “Teaching Algebra and Geometry Concepts by Modeling Telescope Optics.” Mathematics Teacher Mar. 2008: 490-497.

[2]Wassenaan. “Newton’s diverging parabola.” 2dcurves. 2004. 20 Aug. 2008. <http://www.2dcurves.com/cubic/cubicn.html>