Largest Sector in a Right Triangle

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1. Abstract

In this paper, we use Geometry Expressions to investigate the largest sector that can be placed in a right triangle. We make a list of 26 possible cases, and proceed to eliminate ones that cannot possibly be optimal. In the end, we find that the largest such sector is centered at the smallest angle and tangent to the opposite side.

2. Introduction

Many parcels of farmland are divided into squares, generally with half-mile or mile-long sides, and farmers have been fitting irrigation circles into these squares for years, using a technique known as center pivot irrigation. A long sprinkler arm extends from a central pivot, and it slowly rolls around the pivot, watering as it goes. This will generally leave the corners of the square dry, since circles obviously do not tessellate very well, and placing another, smaller circle in the corner would become cost-inefficient. In a slight variation of the circular watering patterns, farmers can also place sectors by limiting the angle that the sprinkler arm is allowed to cover, so that the water only reaches part of the circle. While this arrangement does waste part of the total possible ground coverage for a particular sprinkler system, it does allow for more versatility in sprinkler configurations.

We investigate the best placement for a sector or a circle in a right triangle for two reasons. First, while countless square plots exist, there are occasional roads that cut through fields at awkward angles, resulting in unusual triangles and trapezoids. Sprinkler configurations for squares are usually the same: simply a full circle inscribed in the square. But in the case of a right triangle, placing the largest circle possible—the incircle—is not the optimal placement when taking sectors into consideration. Second, analyzing more complex and realistic cases involving multiple sectors in rectangles and trapezoids is an intimidating task at first. Thus, we begin by simplifying the problem until a solid result is produced. In this case, we reduce the problem to one sector in a right triangle.

3. Solution

We first establish the right triangle. For simplicity, we assume one leg has length 1, and the other leg has length a, as shown in Figure (1).

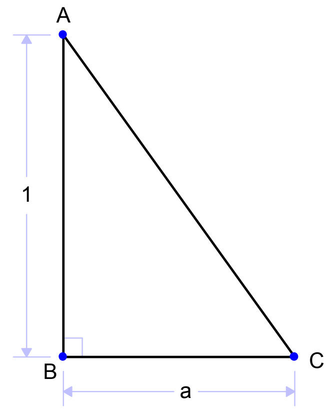


Figure (1)

Without loss of generality, we assume that $AB \geq BC$, therefore $0 < a \leq 1$. This also implies $\angle A \leq \angle C < \angle B$. This accounts for all possible ratios of leg lengths, so any right triangle can be scaled to match. For the purposes of Geometry Expressions, we place B on the origin, and A and C on the positive axes.

We proceed by making a list of all possible cases of a sector in a right triangle. Note that the sector must touch all three sides of the triangle. If it does not, we can translate it toward the third side until it is not touching any sides and then enlarge it from there. There are several ways for the sector to touch a given side of the triangle, shown respectively in Figure (2): they can share one point at a tangent, share one point not at a tangent but on the circle, share one point that is the center of the sector, share two points, share infinite points by having one radius coincide with the side, or share infinite points by having two radii coincide with the side. We define two configurations as *identical* if and only if the cumulative ways in which the triangle touches each of the sides are the same and occur in the same order.

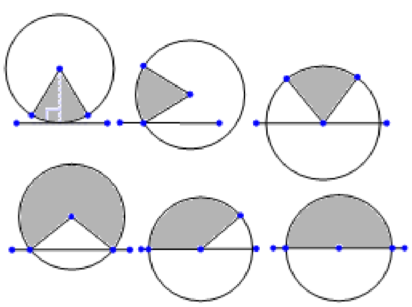


Figure (2)

Note: Since Geometry Expressions cannot draw sectors, we draw them by placing two radii in a circle. They are shaded here for convenience.

However, we exclude cases in which the angle of the sector can be increased without changing the radius or the location of the center; these cases clearly cannot be maximal. For example, in Figure (3), the configuration on the left can be enlarged to the one on the right.

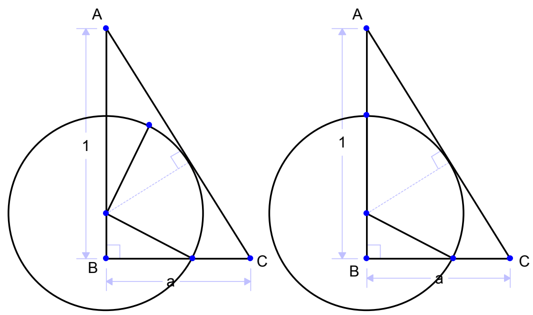
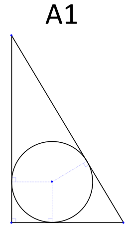


Figure (3)

Lemma. There are 26 non-identical ways to inscribe a sector in a right triangle, which will be denoted {A1, A2,…, A26}.

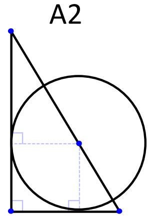
Proof of Lemma. We proceed by examining how many times the sector is tangent to the triangle. If the sector is tangent to the circle at three points, then there is only one configuration, the incircle (A1).

A1

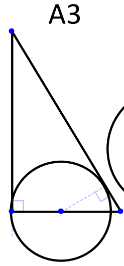


Assume the sector is tangent to the circle at two points. If the center is on a side of the triangle, then we have three possible semicircles (A2-A4). If the center is not on the triangle, then we have three possible sectors (A5-A7).

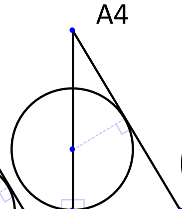
A2



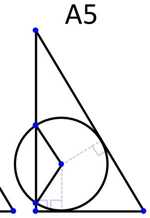
A3



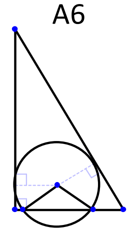
A4



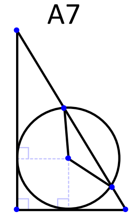
A5



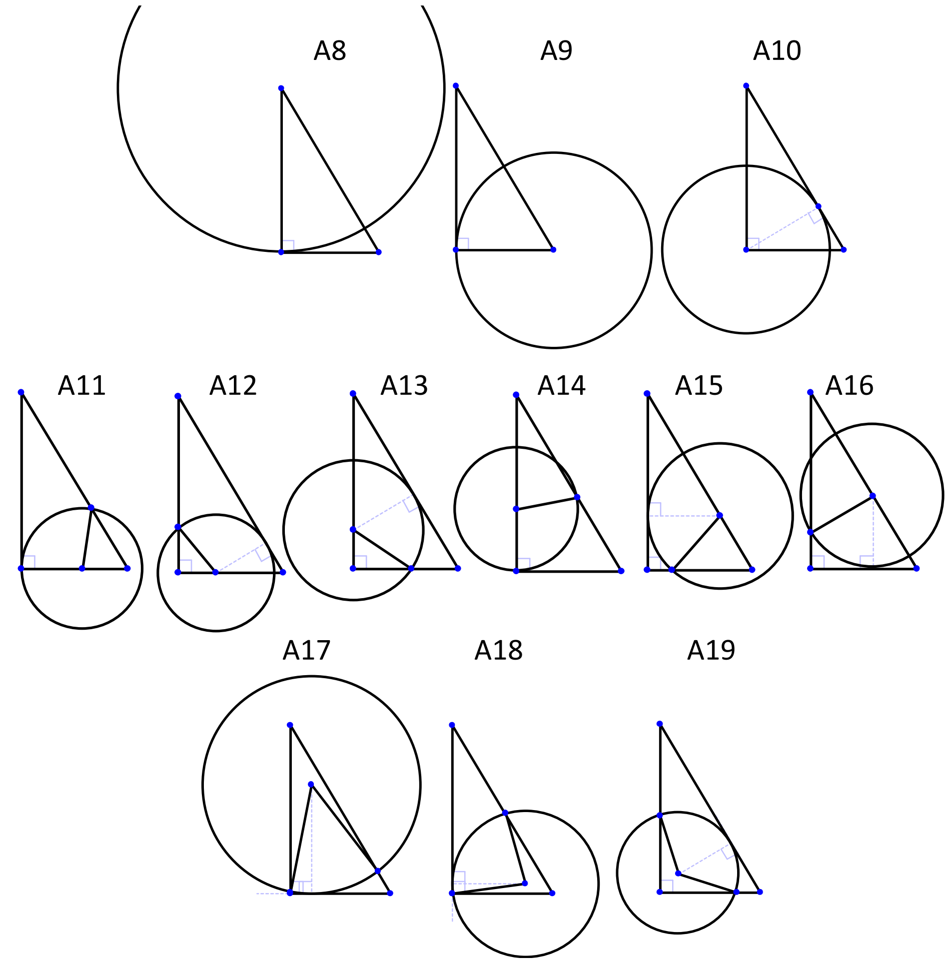
A6



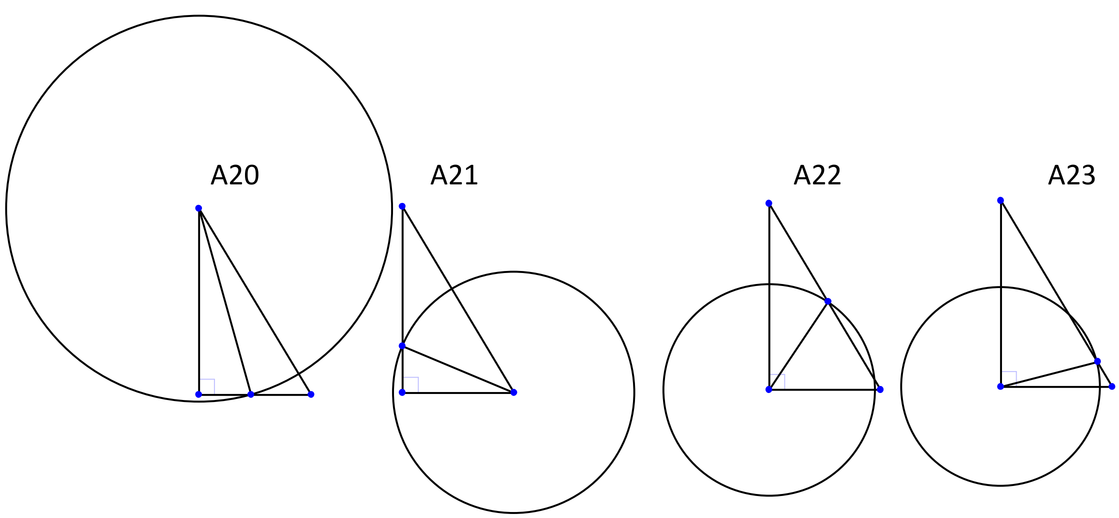
A7



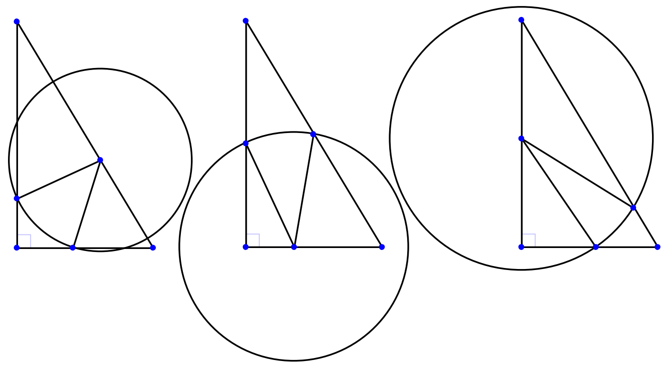
Assume the sector is tangent to the circle at one point. If the center is on a vertex of the triangle, then we have three possible sectors (A8-A10). If the center is on a side of the triangle (excluding vertices), then we have six possible sectors, all with one radii along the side (A11-A16). If the center is not on the triangle, then we have three possible sectors (A17-A19).



Finally, assume the sector is not tangent to the circle. Then, the center must lie on a side to achieve the condition of touching all three sides of the triangle. If it is on a vertex, there are four possible sectors (A20-A23). If it is on a side and not on a vertex, then there are three possible sectors (A24-A26). □



A24 A25 A26



We now proceed by eliminating cases that cannot be optimal.

Eliminating A24, A25, and A26. A25 and A26 differ only in the way the sector is oriented with respect to the leg lengths. Thus, it is sufficient to prove that the appropriate sector for an arbitrary ratio of legs is not optimal. We proceed by contradiction; assume that a given sector DEF is optimal (Figure (4)).

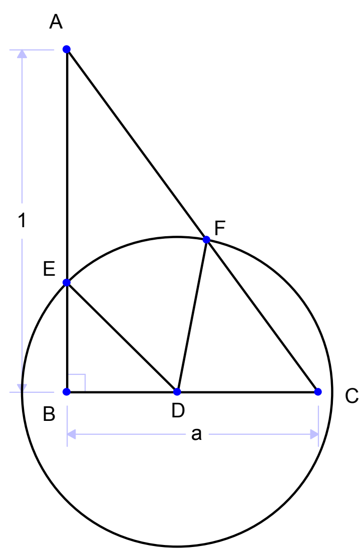


Figure (4)

We take the locus of F as D slides along BC and E slides along AB. Since DEF is a fixed triangle, this locus is an ellipse centered at the origin. In Figure (5), the red line is the locus, and the hypotenuse is hidden. We see that no matter where the sector is, it will always be possible to move point E into the interior of the triangle. (The only exception to this is if point E starts at M or L; however, then the case has turned into A22 or A23.) Thus, once the sector is no longer touching the hypotenuse, we can enlarge it, contradicting our assumption that the original sector was maximal. Note that the arc $\widehat{DE}$ will not move outside of the triangle. If it becomes tangent to the triangle, we can stop there and increase the sector's angle, thereby turning the case into A12 or A13.

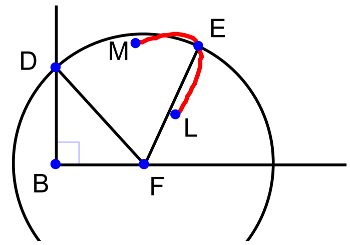


Figure (5)

We use a similar argument for A24. In Figure (6), assume sector DEF is optimal. Then, sliding points D and E along the appropriate lines, we can move point F so that it is no longer touching the third side, which is hidden in the diagram. At that point, we can enlarge the sector, contradicting our assumption that the original sector was optimal.

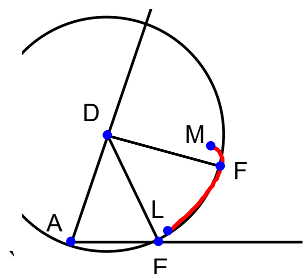


Figure (6)

Eliminating A22 and A23. We draw A8 over A22, and show that A22 is always completely covered. These are both represented in Figure (7). Sector ABG represents A8, and sector BEF represents A22 with maximal sector angle (clearly, any smaller sector angle will be fully covered if the sector with maximal angle is covered). We have $AG = AB = 1 > AF$ (because AB is the hypotenuse of triangle AFB), so BF will always lie above arc BG, and therefore A22 is always within A8.

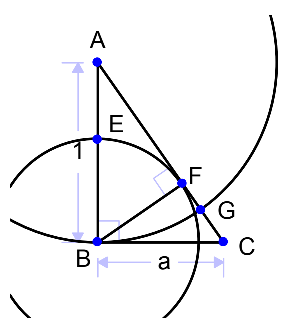


Figure (7)

A similar argument holds for A23; it is always covered by A9.

Eliminating A20 and A21. We begin with A20 and A21, shown in Figures (8a) and (8b), respectively. We define the angles of the sectors to be m and n, and Geometry Expressions calculates the radii as shown. The areas of the sectors are thus $\frac{m(1+a^2)}{2(a*\sin(m)+\cos(m))^2}$ and $\frac{a^2 n(1+a^2)}{2(a*\cos(n)+\sin(n))^2}$.

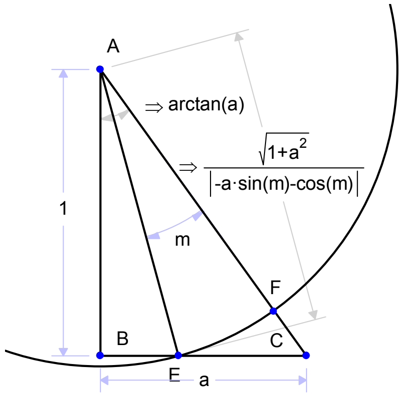


Figure (8a)

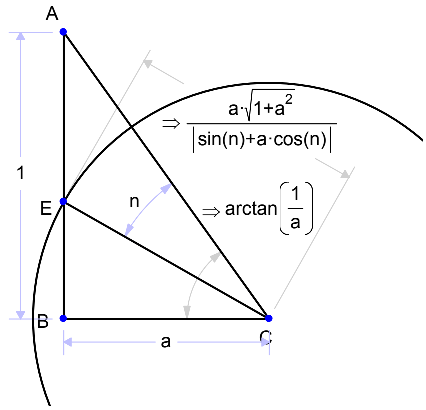


Figure (8b)

We plot the area of the sector for A20 in Mathematica, shown in Figure (9). We see that the curve is always increasing for a given value of a. This can be verified by deriving the area expression with respect to m, and observing that it is positive for all (m, a) within our desired domain.

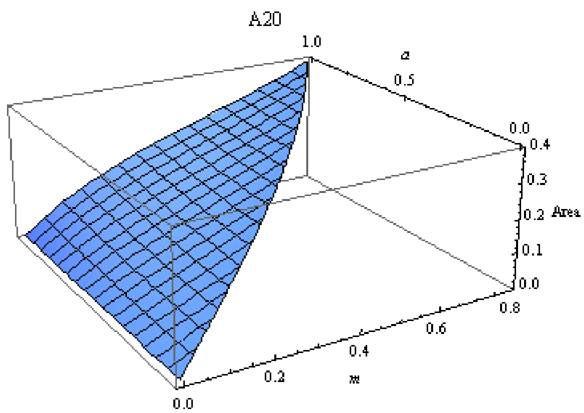


Figure (9)

Thus, for a given triangle, A20 is maximized when the angle of the sector is maximized; this is A8. Now, we graph A21 (Figure (10a)), and A21 against the maximum of A20 (Figure (10b)). Although is it not clear if A21 is always increasing at smaller values of a, we see that A8, or maximal A20, is in any case always better than A21.

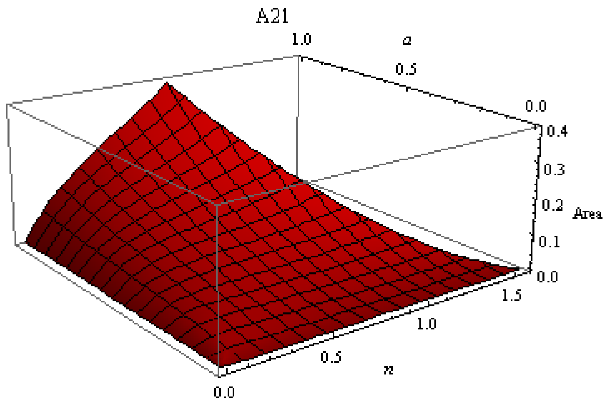


Figure (10a)

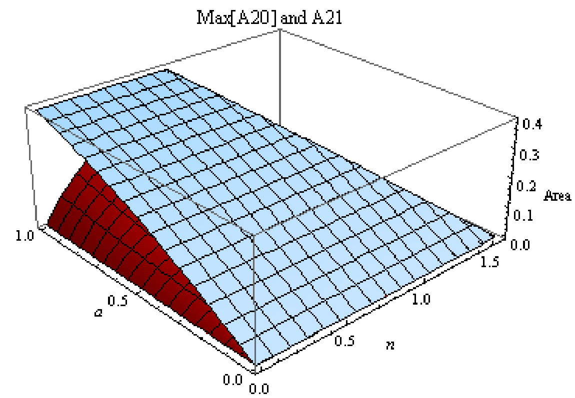


Figure (10b)

Eliminating A17, A18, and A19. A17 and A18 only differ in the orientation of the sector with respect to the leg lengths, so it is sufficient to prove that the appropriate sector is not optimal for an arbitrary ratio of leg lengths. We proceed by contradiction. Assume that a given sector DEF is optimal (Figure 11); the hypotenuse is hidden. We find the locus of F as E slides along its vertical line. Since the hypotenuse intersects the locus at F, we will always be able to slide E so that F lies strictly inside the triangle. The only instance when we could not do this is if the locus is tangent to the hypotenuse and lies outside of the triangle, which will not happen because the locus will always be concave toward B. Thus, once the sector no longer touches the hypotenuse, we can enlarge it, contradicting the assumption that it is maximal.

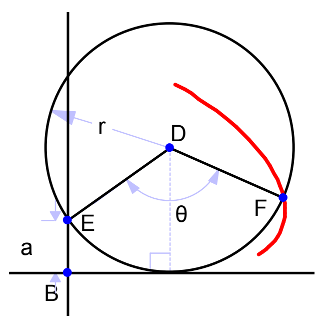


Figure (11)

We use a similar argument for A19, shown in Figure (12).

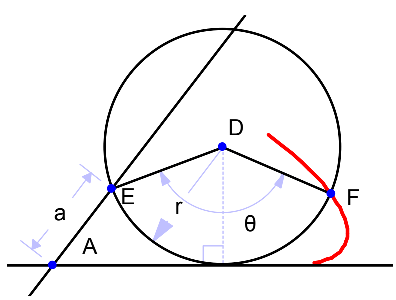
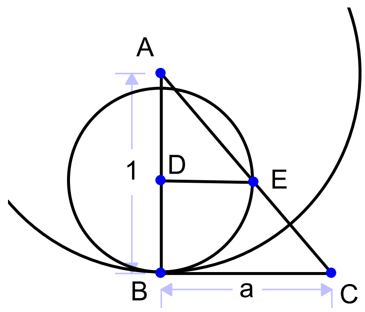


Figure (12)

Eliminating A11 and A14. We draw A8 on A11 (Figure (13a)) and A9 on A14 (Figure (13b)). We see A11 and A14 will always be entirely covered by A8 and A9, respectively; thus they cannot be optimal.



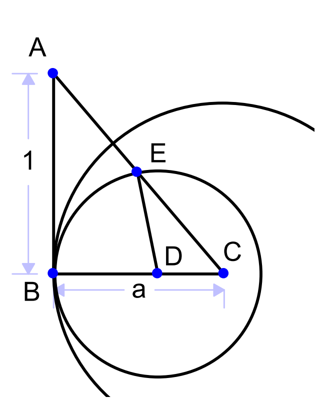
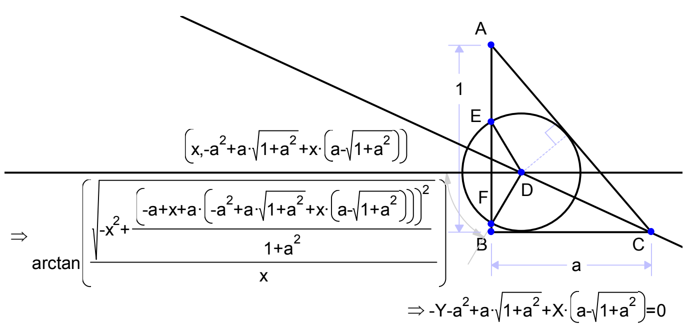
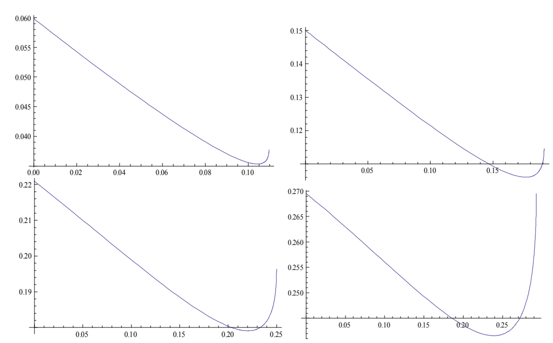


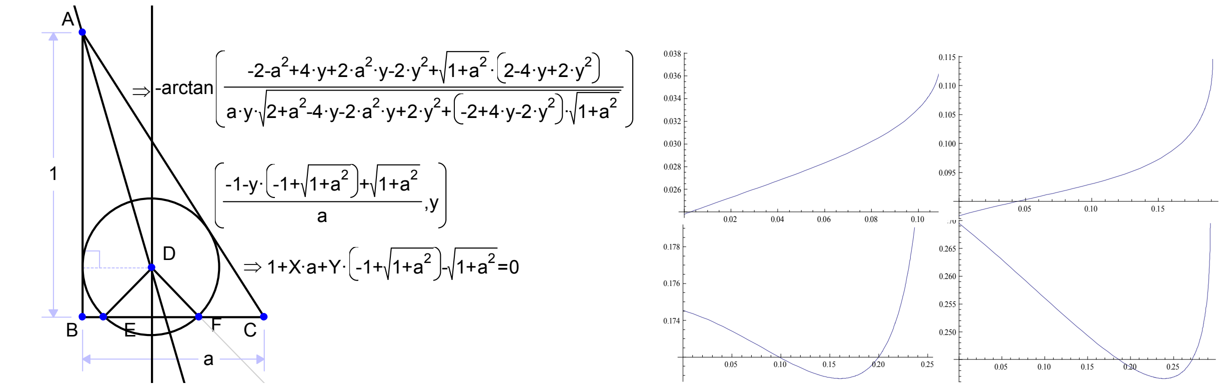
Figure (13a) Figure (13b)

Eliminating A5, A6, A7, A12, A13, A15, and A16. A5, A6, A7, A12, A13, A15, and A16 are all eliminated using the same method. We graph the area of the sector, which can be expressed in two variables, and it is clear that the maximal area occurs at one of the extremes, all of which are other cases with both radii coinciding with the triangle's sides. However, since 3D graphs are difficult to interpret on paper, we present cross-sections of the 3D graph at $\mathrm{a} = \{.25, .5, .75, 1\}$ as a summary of the entire graph.

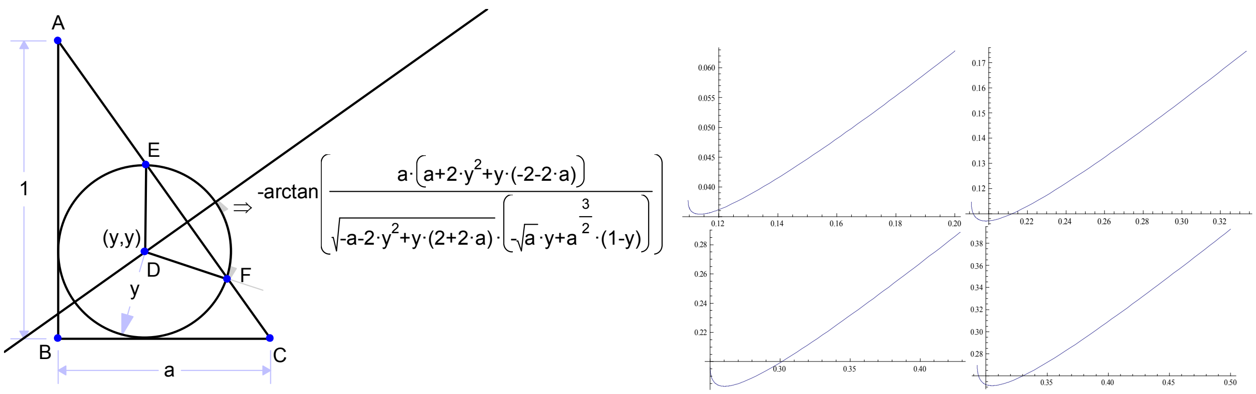




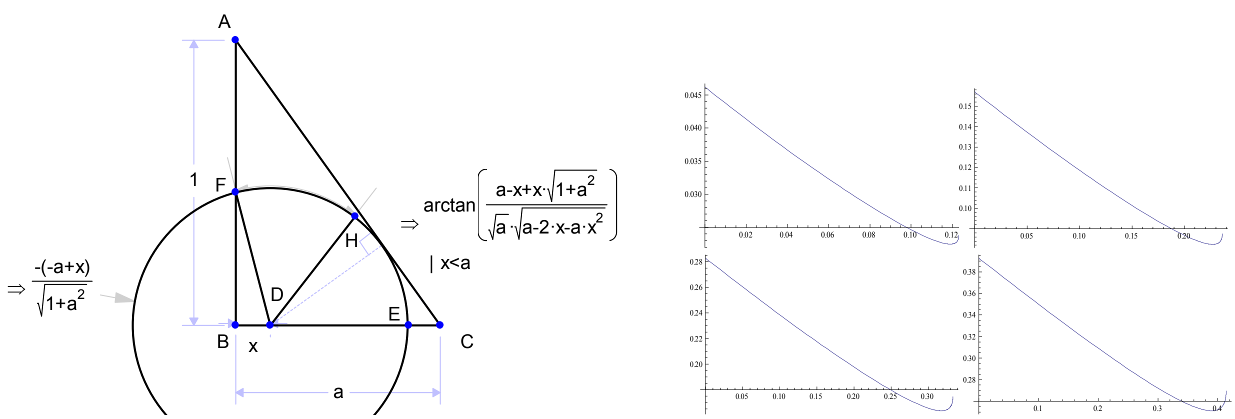
A5. We define the center of the sector by its x-coordinate x, which is the x-axis variable in the graphs. We see that maximal area always occurs at minimal x, which means that A4 is always better than A5.



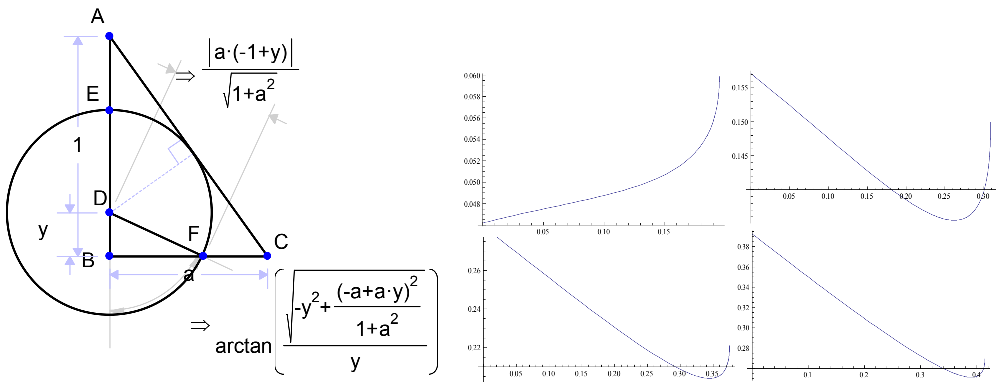
A6. We define the center of the sector by its y-coordinate y, which is the x-axis variable in the graphs. We see that maximal area occurs at either maximal or minimal y, which means that either A1 or A3 will always be better than A6.



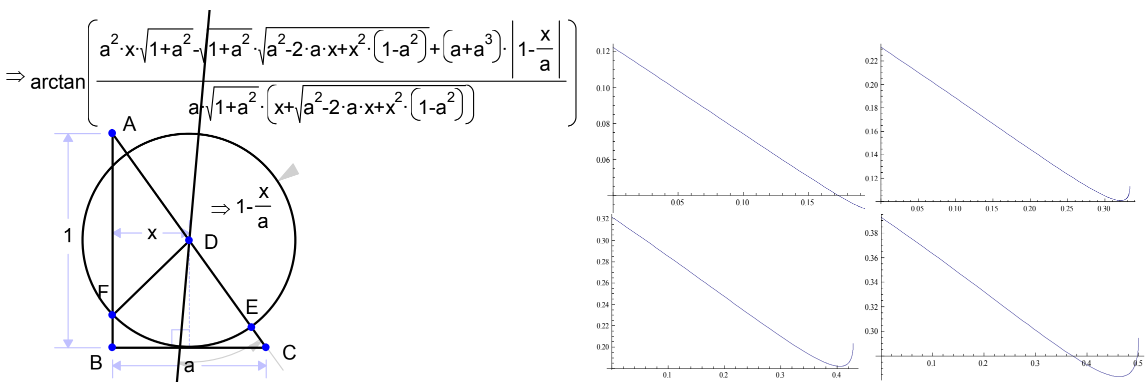
A7. We define the center of the sector by its y-coordinate y, which is the x-axis variable in the graphs. We see that the maximal area occurs at maximal y, which means that A2 will always be better than A7.



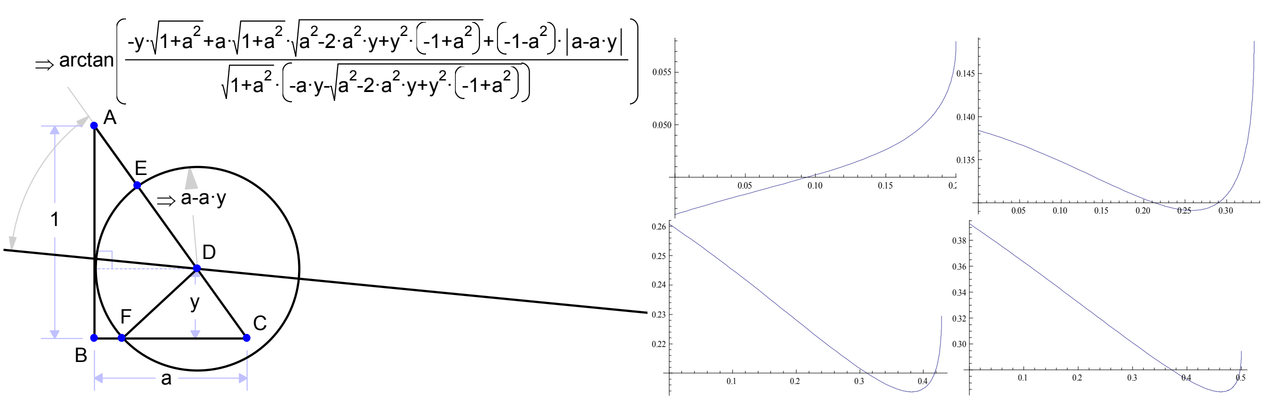
A12. We define the center of the sector by its x-coordinate x, which is the x-axis variable in the graphs. We see that the maximal area occurs at minimal x, which means that A10 will always be better than A12.



A13. We define the center of the sector by its y-coordinate y, which is the x-axis variable in the graphs. We see that the maximal area occurs at either maximal or minimal y, which means that either A4 or A10 will always be better than A13.



A15. We define the center of the sector by its x-coordinate x, which is the x-axis variable in the graphs. We see that the maximal area occurs at minimal x, which means that A8 will always be better than A15.



A16. We define the center of the sector by its y-coordinate y, which is the x-axis variable in the graphs. We see that the maximal area occurs at either maximal or minimal y, which means that either A9 or A2 will always be better than A16.

The remaining cases. We have A1, A2, A3, A4, A8, A9, and A10 left; all other cases either can be enlarged to one of these. Note that these are all fixed for a given triangle, so their areas only depend on the value of a. Thus, we can generate expressions for the area and use a simple graph to evaluate the remaining cases (Figure (14)).

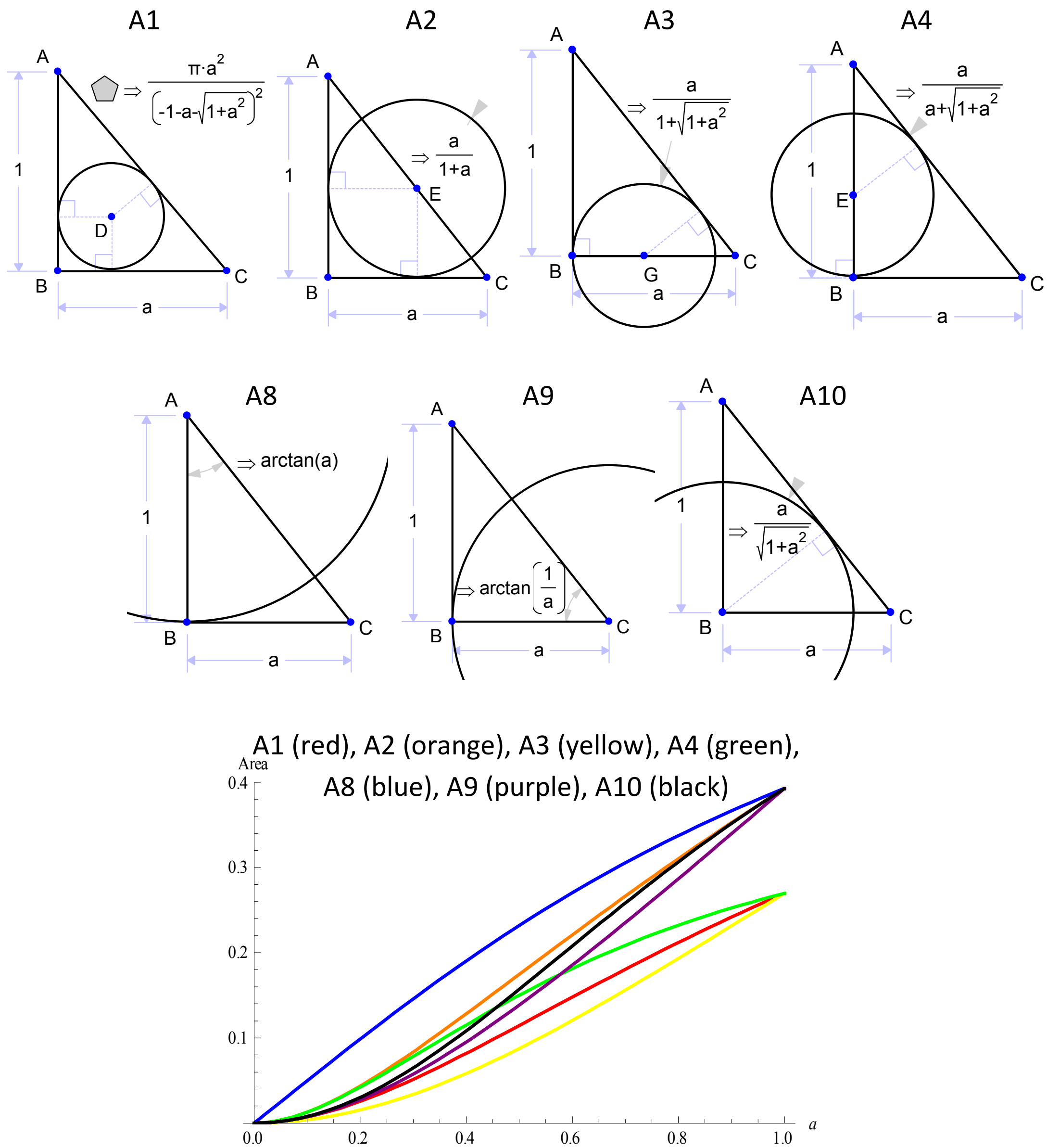


Figure (14)

We see that the blue line, or A8, is always on top. Thus, in a right triangle, the largest sector will be one centered at the vertex of the smallest angle and is tangent to the opposite side.

Remark. It is interesting that all of these seven areas converge to one of two areas at $a = 1$, the isosceles case. The three sectors centered at the vertices and the semicircle on the hypotenuse have area $\frac{\pi}{8} \approx .393$, while the incircle and other two semicircles have area $\frac{\pi}{\left(2-\sqrt{2}\right)^2} \approx .270$. Also, it would be natural to assume from this that the sector at the smallest angle is the largest for all triangles. However, this is not the case; for example, in an isosceles triangle with base angles $\frac{2\pi}{9}$, the sector at the largest angle is actually bigger.

4. Real-life Analysis

In Lubbock, Texas, highway 194 runs diagonally through a square field, and a farmer has placed three sectors inside this triangle. Although one sector centered at a vertex is likely too big for practical reasons, we present an analysis of the farmer's sectors in comparison to the ideal.

We superimpose a Geometry Expressions diagram onto the image, and use the numerical calculator to measure lengths and angles. Note that all lengths are relative; they can all be scaled up or down.

In the field, the top sector has angle 3.95552 and radius .0627664. The middle one has angle 4.689922 and radius .1061178. The last one has angle 3.14159 and radius .109768. Thus, the total area is .0531247. The ideal sector, centered at the top angle, has angle .7823217 and radius .380886, and thus area .0567473.

The farmer comes fairly close to the ideal coverage. Note, however, that his configuration can be directly improved if we assume the omitted part of the middle sector is a quartercircle, and the triangle is isosceles. The left half of the semicircle is equal to the omitted part, so we can make the middle sector a whole circle. Applying the proof in this paper to the right triangle shown, the current sector is equal in area to a sector centered at the far right angle. However, this sector can be expanded beyond the right triangle until it is tangent with the full circle.

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