Light Caustics

If you look at light reflected within a cylinder, you see a bright curve caused by the light being concentrated into a specific region. This curve is called a caustic.

You can see this curve readily if you put a shiny wedding ring on a piece of paper. But all sorts of cylindrical objects will generate caustics, including coffee mugs, pots and pans, and in the pictures below, a piece of perspex tubing.

Figure 1: Caustic inside a cylinder with the light part of the way from the circumference to the center.

When the light source is outside the cylinder, as we will see, the caustic rapidly becomes effectively equivalent to one generated from an infinitely distant light. Here we want to look at the shapes generated by a light source inside the cylinder. For this we need to find a big enough cylinder and a small enough light. With this kind of apparatus, we can generate an interesting family of caustic curves:

In order to make quantitative observations on the curve, you need to create some circular graph paper. You can do this in Geometry Expressions itself. Draw 10 concentric circles, then set the radius of the first circle to be r, the radius of the second to be 2*r, the third to be 3*r, etc. Print it out and see how much bigger or smaller than the cylinder your 10th circle is. Then adjust r accordingly and print out again.

Light on the circumference

Figure 2 shows the caustic when the light source is placed at the circumference of the cylinder. Using the circular graph paper, estimate how far from the center the cusp is in this situation.

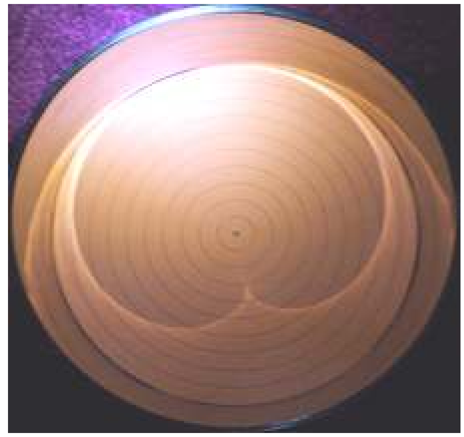

Figure 2: Caustic curve with the light at the radius of the cylinder

We can model the behavior of a single beam of light in Geometry Expressions. First we create a circle AB, and constrain its center to be (0,0) and its radius to be 1. We then constrain the parametric location of B on the curve to be t. This has the effect of specifying the AB to be angle t (in radians) from the x axis. Geometry Expressions allows you to reflect in a line, but not in a curve. However reflection of light in a cylinder is equivalent to reflection in the tangent to the cylinder. Hence we draw a line through B and constrain it to be tangent to the circle. We now create point C, constrain its location to be (0,-a), and draw an infinite line through C and B. We complete the model by reflecting BC in the tangent line.

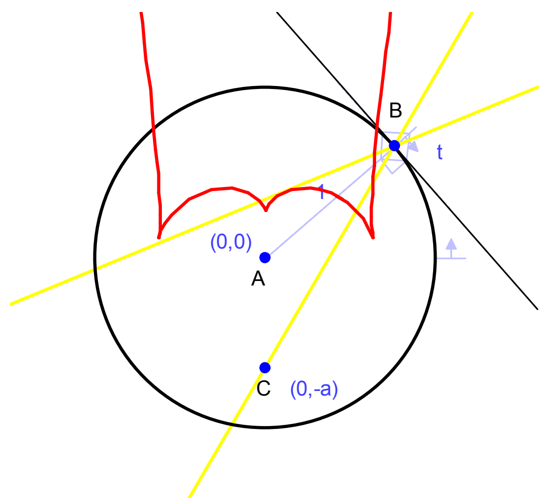

Figure 3: Geometry Expressions model of a beam of light reflected in a cylinder.

To display the caustic curve, you need to create the envelope of the reflected line. This is done by selecting the reflected line and using the Locus tool. With the caustic curve displayed, try dragging C closer or further away from the center of the circle. Do the curve shapes match those you observed?

We are interested in the location of the central cusp. First in the situation where a = 1. Change the coordinates of C to (0,-1), and then display the parametric equation of the caustic curve. (Fig 4)

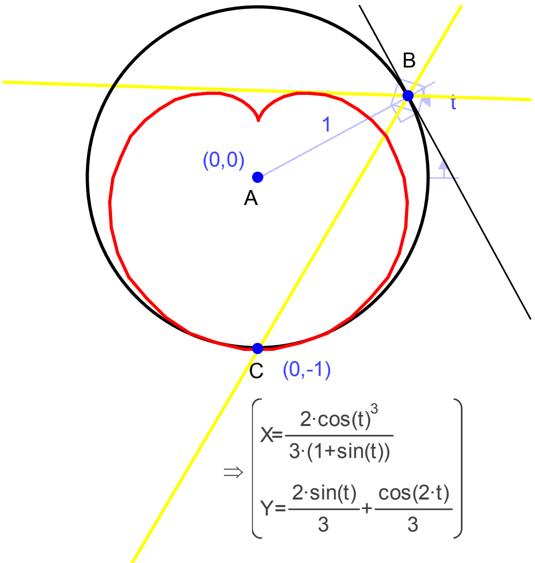

Figure 4: Parametric equation of the caustic with light source on the circumference

What value will t have when the reflected ray passes through the cusp (remember t is the angle in radians of AB)?

What is the y value of the curve for that value of t?

Does that result match your observations?

Cusp on the Circumference

As the light source moves towards the center of the cylinder, the caustic curve is no longer fully contained in the cylinder. However, at some point, it re-enters to give the shape in Figure 5. Again, using the circular graph paper, estimate the location of the central cusp at this point.

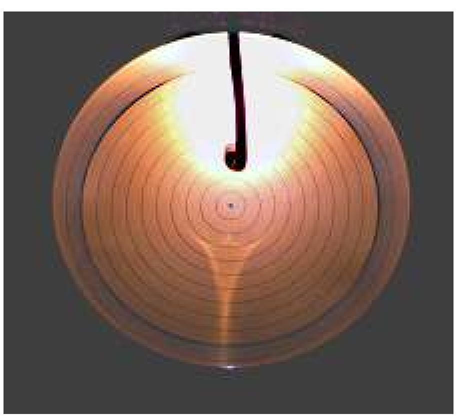

Figure 5: The caustic when the far cusp touches the circumference.

In Geometry Expressions, change the coordinates of C back to (0,-a) and drag C towards the center of the circle until the second central cusp appears:

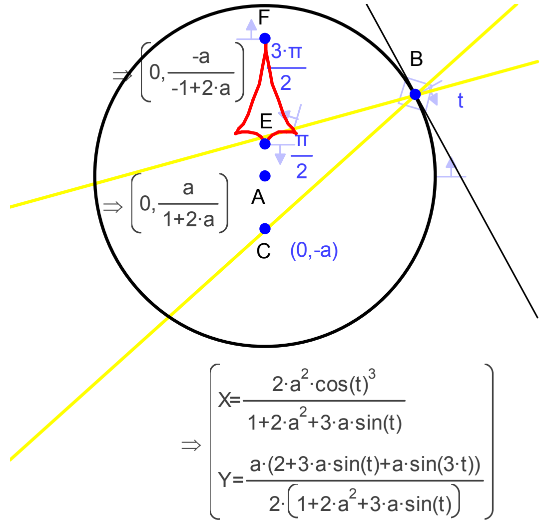

Figure 6: Curve equations for general a. Points E and F are put on the curve and constrained to be at parametric locations π/2 and 3π/2

In figure 6, we have put points E and F on the curve and constrained them to be parametric distance π/2 and 3π/2 along the curve. We have displayed the coordinates of E and F

What value of **a** will put F at the circumference?

In which case, where will E lie?

Does this match your observations?

Further Questions

As **a** gets larger and larger ("tends to infinity"), where does the cusp at E go?

Can you verify this experimentally? (The room lighting will be effectively at infinite distance, unless you happen to have placed your tube directly under it.)

For a curve (X(t),Y(t)), cusps occur where the derivatives of X with respect to t and Y with respect to t simultaneously vanish.

If you have access to a computer algebra system, you might like to try to determine the parameter value (t) of the other caustics.

Can you interpret this geometrically?

Can you derive the coordinates of the cusps?

Can you display the locus of the cusps as a varies?