Loci
Here are some locus examples
Example 121: Circle of Apollonius

The Circle of Apollonius is the locus of points the ratio of whose distance from a pair of fixed points is constant:

\[ \Rightarrow 2 \cdot X \cdot a^2 + X^2 \cdot (-1 + k^2) + Y^2 \cdot (-1 + k^2) = 0 \]

What is the center and radius?
**Example 122: A Circle inside a Circle**

Points D and E are proportion $t$ along the radii AD and AC of the circle centered at the origin and radius $r$. The intersection of CD and DE traces a circle.

Show that it goes through the origin. What is the center of the circle? What is its radius?
Example 123: Another Circle in a Circle

More generally if D is proportion s along AC, we have the following circle:

\[ r^2 s^2 t^2 + r^2 s^2 t^2 + 2r^2 s^2 t^2 + X^2 \left( 1 - 2s + s^2 t^2 \right) + Y^2 \left( 1 - 2s + s^2 t^2 \right) + Z^2 \left( 2rs + 2rs + 2rs + 2rs + 2rs + 2rs \right) = 0 \]

What is the center of this circle?

Can we find the radius of this – perhaps by copying the expression into an algebra system and working on it there?
Here is one approach, in Maple. First we substitute $Y=0$, then solve for $X$ to determine the $x$ intercepts of the circle. The radius can be found by subtracting these and dividing by 2.

```maple
> subs(Y=0,-s^2*r^2+2*t*s^2*r^2+t^2*r^2-2*t^2*s*r^2+(-2*t*s+1+t^2*s^2)*X^2+(-2*t*s+1+t^2*s^2)*Y^2+(-2*t*r+2*t*s*r+2*t^2*s*r-2*t^2*s^2*r)*X = 0);
-s^2 r^2 + 2 t s^2 r^2 + t^2 r^2 - 2 t^2 s r^2 + (-2 t s + 1 + t^2 s^2) X^2 + (-2 t r + 2 t s r + 2 t^2 s r - 2 t^2 s^2 r) X = 0

> solve(%,X);
\( r (-t + s)/ts - 1, r (-t - s + 2 t s)/ts - 1 \)

> (r*(-t+s)/(t*s-1)- r*(-t-s+2*t*s)/(t*s-1))/2;
\( r (-t + s)/2(t s - 1) - r (-t - s + 2 t s)/2(t s - 1) \)

> simplify(%);
\( -r s (-1 + t)/ts - 1 \)
Example 124: Ellipse as a locus
Here is the usual string based construction of an ellipse foci (-a,0) (a,0):

\[ \Rightarrow -L^4 + 4L^2 \cdot Y^2 + 4L^2 \cdot a^2 + X^2 \left( 4L^2 - 16 \cdot a^2 \right) = 0 \]
Example 125: Archimedes Trammel

A mechanism which generates an ellipse is Archimedes Trammel. The points C and E are constrained to run along the axes, while the distance between them is set to $a-b$. We trace the locus of the point D distance $b$ from E along the same line. This gives an ellipse with semi-major axes $a$ and $b$:

\[ \Rightarrow y^2 \cdot a^2 + x^2 \cdot b^2 - a^2 \cdot b^2 = 0 \]
Example 126: An Alternative Ellipse Construction
Here is a construction (ascribed to Newton) which builds the ellipse from concentric circles radius equivalent to the semi major axes

\[ Y^2 \cdot a^2 + X^2 \cdot b^2 - a^2 \cdot b^2 = 0 \]
Example 127: Another ellipse
This time take a circle and a point, and the location of all points equidistant from the circle and the point:

\[ 4Y^2r^2 + 4a^2r^2 - r^4 + X^2(-16a^2 + 4r^2) = 0 \]
**Example 128: “Bent Straw” Ellipse Construction**

Here is another ellipse construction. Geometrically observe that the semi major axes are $x-a$ and $x+a$. Can you verify this from the algebraic expression?

\[
-a^4 + 2ab^2 + b^4 + x^2 \left( a^2 - 2ab + b^2 \right) + y^2 \left( a^2 + 2ab + b^2 \right) = 0
\]
Example 129: Similar construction for a Hyperbola
If we do a similar construction, with the generating point outside the circle, we get a hyperbola:

\[ 4 \cdot r^2 \cdot r^2 + 4 \cdot a^2 \cdot r^2 \cdot r^2 - r^4 + X^2 \cdot (-16 \cdot a^2 + 4 \cdot r^2) = 0 \]
Example 130: Parabola as locus of points equidistant between a point and a line

Here is the equation of the parabola which is the locus of points equidistant from the point (-a,0) and the line X=a:

\[ y^2 + 4X \cdot a = 0 \]
**Example 131: Squeezing a circle between two circles**

Take a circle radius 2a centered at (a,0) and a circle radius 4a centered at (-a,0). Now look at the locus of the center of the circle tangent to both.

\[8X^2 + 9Y^2 - 72a^2 = 0\]

It’s an ellipse. From the drawing we can see that the semi major axis in the x direction is 3a. What is the semi major axis in the y direction?
Example 132: Rosace a Quatre Branches

This example comes from the September 2003 edition of the Casio France newsletter.

A line segment of length \( a \) has its ends on the x and y axes. We create the locus of the orthogonal projection of the origin onto this segment. Apparently this curve was studied in 1723-1728 by Guido Grandi.

\[
X^6 + 3X^4Y^2 + 3X^2Y^4 + Y^6 - X^2Y^2 - a^2 = 0
\]
Example 133: Lemniscate

Given foci at (-a,0) and (a,0), the lemniscate is the locus of points the product of whose distance from the foci is $a^2$:

\[(-a,0) \quad (a,0)\]

\[t\]

\[t\]

\[a^2\]

\[a^2\]

\[=0\]

\[\Rightarrow -X^4\cdot 2\cdot X^2\cdot Y^4 + 2\cdot X^2\cdot a^2\cdot Y^2\cdot a^2 = 0\]
Example 134: Pascal's Limaçon
Named after Etienne Pascal (1588-1651), father of Blaise.

\[ X^4 + 2X^2 \cdot Y^2 + Y^4 - a^2 \cdot 2X^3 \cdot b + 2X \cdot a^2 \cdot b + b^2 + X^2 \cdot (a^2 + b^2) = 0 \]
Example 135: Kulp Quartic
Studied by, you guessed it – Kulp, in 1868:

\[ x^2 \cdot y^2 \cdot r^2 \cdot t^4 = 0 \]
Example 136: The Witch of Agnesi
Named after Maria Gaetana Agnesi (1748)

\[ X^2 \cdot Y + X^2 \cdot r + 4 \cdot Y^2 \cdot r - 4 \cdot r^3 = 0 \]
Example 137: Newton's Strophoid

\[ x^2 - y^2 - xy^2 - y^3 = 0 \]
Example 138: **MacLaurin’s Trisectrix and other Such Like**

A cubic derived from the intersection of two lines rotating at different speeds.

\[-X^3 \cdot Y^2 + 3 \cdot X^2 \cdot a \cdot Y^2 \cdot a = 0\]
A similar construction can give a range of other curves. For example, a hyperbola:

\[ -3X^2 + Y^2 + 2X \cdot a = 0 \]
Example 139: Trisectrice de Delange

\[ X^2Y^2 + Y^4 - 4X^2a^2 - 4Y^2a^2 + 4a^4 = 0 \]

A \((0,0)\)

B

C \((2,0)\)

D

E

2·t

2·t

a

t
Example 140:  “Foglie del Suardi”

Here is a cubic which can be drawn by a mechanism consisting of intersecting a particular radius with a particular chord of a circle.

\[-X^3 - X^2 \cdot Y^2 - 2 \cdot X^2 \cdot a - 2 \cdot Y^2 \cdot a - X \cdot a^2 = 0\]
Example 141: A Construction of Diocletian

\[
\begin{align*}
X &= \frac{2 \cdot t^2}{4 + t^2} \\
Y &= \frac{t^3}{4 + t^2}
\end{align*}
\]

\[\Rightarrow -X^3 + 2 \cdot Y^2 - X \cdot Y^2 = 0\]

Segment CF is defined to be congruent to GE. Diocletian used this construction to define a cubic curve.
Example 142: **Kappa Curve**

Studied by Gutschoven in 1662, the locus of the intersection between a circle and its tangent through the origin as the circle slides up the y-axis:

\[ X^4 + X^2Y^2 - Y^2r^2 = 0 \]
Example 143: Kepler's Egg

An egg shape defined by projecting B onto AC, then back onto AB then back onto AC:

$$X^4 + 2X^2Y^2 + Y^4 - X^3a = 0$$
Example 144: Cruciform Curve

\[ -X^2 \cdot Y^2 + X^2 \cdot Y^2 = 0 \]

this curve can be rewritten in the form:

\[ \frac{1}{x^2} + \frac{1}{y^2} = 1 \]
**Example 145: Locus of centers of common tangents to two circles**

We take the locus as the radius \( r \) of the left circle varies. The midpoints of all four common tangents lie on the same fourth order curve:

\[
4 X^4 + 8 X^2 Y^2 + 4 Y^4 - 12 X^2 a^2 + 4 a^2 + s^2 X Y^2 + (4 a^2 - 4 s^2) X^2 + (-6 a^3 + 4 a^2 s^2) = 0
\]

We can use Maple to solve for the intersections with the x axis:

```maple
> subs(Y=0, 4*X^4 + 8*X^2*Y^2 + 4*Y^4 - 12*a*X^3 - 12*a^2*X^2 + a^4 - s^2*a^2 + (4*a^2 - 4*s^2)*Y^2 + (13*a^2 - 4*s^2)*X^2 + (-6*a^3 + 4*a^2*s^2)*X)

4 X^4 - 12 a X^3 + a^4 - s^2 a^2 + (13 a^2 - 4 s^2) X^2 + (-6 a^3 + 4 s^2 a) X

> solve(%, X);
```
\[ a - s, a + s, \frac{1}{2} a, \frac{1}{2} a \]
Example 146: Steady Rise Cam Curve

Assuming a Flat Plate reciprocating follower, here is the cam curve for a linear rise of \( k \cdot t + c \). This is the Envelope of the line BE.

\[
\begin{align*}
X &= -k \cdot \sin(t) + (c + k \cdot t) \cdot \cos(t) \\
Y &= c \cdot \sin(t) + k \cdot t \cdot \sin(t) + k \cdot \cos(t)
\end{align*}
\]
Example 147: Oscillating Flat Plate Cam

Here is a cam curve for an oscillating flat plate cam follower, where the follower rise is linear in the cam angle: \( \text{rise} = u + tv \)

\[
\begin{align*}
X &= a \cdot \frac{-1 + 2 \cdot \cos(tv)}{2} - 1 + 2 \cdot \cos(u) \\
Y &= a \cdot \frac{-1 + 2 \cdot \cos(tv)}{2} \cdot \sin(t) + 2 \cdot \sin(u) \cdot \cos(t) \cdot \cos(u) + 2 \cdot \cos(t) + 2 \cdot \sin(t) \cdot \sin(u) \cdot \cos(u) \cdot \sin(t \cdot v) \cdot \cos(t \cdot v) \\
&\quad \cdot \sin(t) + 2 \cdot v \cdot \sin(t) \\
&\quad 2 \cdot (1 + v)
\end{align*}
\]

\( b > 0 \)

Diagram showing the cam curve and the relationship between the points A, B, C, D, and E.
Example 148: A Cam Star

Based on the previous model, let’s take the simple case where the follower angle is twice the cam angle:

\[
\begin{align*}
X &= \frac{3a \cdot \cos(t) + a \cdot \cos(3t)}{2} \\
Y &= \frac{3a \cdot \sin(t) - a \cdot \sin(3t)}{2}
\end{align*}
\]
Can we get an implicit definition of the curve? Yes.

\[ x^6 + 3x^4y^2 + 3x^2y^4 + y^6 - 12x^4a^2 + 84x^2a^2y^2 - 12y^4a^2 + 48x^2a^4 + 48y^2a^4 - 64a^6 = 0 \]
Example 149: Ellipse as Envelope of Circles
Take the envelope of the circles whose centers lie on the x-axis and which have extrema which lie on the unit circle. We find it is an ellipse:

\[ -2X^2 + 2Y^2 = 0 \]
Example 150: **Hyperbola as an envelope of circles**

Take the envelope of a family of circles centered on a line and whose radius is an eccentricity times the distance from a focus.

\[ e^2 + y^2 - e^4 + x^2 (1 + e^2) = 0 \]
Example 151: Hyperbola as an Envelope of Lines

We take the envelope of the perpendicular bisectors of the line CD as C traverses the circle AB.

\[ \Rightarrow \ -a^4 + 4r^2 + 2a^2 r^2 + r^4 + X^2 \left( -4a^2 + 4r^2 \right) + X \left( 4a^3 - 4ar^2 \right) = 0 \]

The result is a hyperbola with foci A and B.

What happens if D lies inside the circle?
Example 152: Caustics in a cup of coffee

The Nephroid curve generated by reflecting a set of parallel rays in a circle, and then taking the envelope of the reflected rays:

\[
64X^6 + 192X^4Y^2 + 192X^2Y^4 + 64Y^6 - 48X^4r^2 - 96X^2Y^2r^2 - 48Y^4r^2 - 15X^2r^4 + 12Y^2r^4r^6 = 0
\]
Example 153: A Nephroid by another route

The envelope of the circles whose centers lie on a circle and which are tangential to the diameter form the same type of curve:

\[ 4X^6 + 12X^4Y^2 + 12X^2Y^4 + 4Y^6 - 12X^4a^2 + 24X^2Y^2a^2 - 12Y^4a^2 + 12X^2a^4 - 15Y^2a^4 - 4a^6 = 0 \]
Example 154: Tschirnhausen’s Cubic

Studied by Ehrenfried Tschirnhausen in 1690, this is the caustic of a set of parallel rays perpendicular to the axis of a parabola:

\[ 108 \cdot X^2 - 81 \cdot Y + 72 \cdot Y^2 - 16 \cdot Y^3 = 0 \]
Example 155: Cubic Spline

This diagram shows an algorithm for constructing the cubic spline from its control points:

\[
\begin{align*}
X &= x_0 + 3t^2x_2 + 3t^3x_3 - 3t^2x_1 + 3t^3x_4 + 3t^2x_3 - 3t^3x_2 + t^3x_3 \\
Y &= y_0 + 3t^2y_2 + 3t^3y_3 - 3t^2y_1 + 3t^3y_4 + 3t^2y_3 - 3t^3y_2 + t^3y_3
\end{align*}
\]

Point E is proportion \( t \) along the line AB. Point F is proportion \( t \) along BC. Point G is proportion \( t \) along CD. Point H is proportion \( t \) along EF. Point I is proportion \( t \) along FG. Point J is proportion \( t \) along HI. The spline curve is the locus as \( t \) runs from 0 to 1.
**Example 156: A Triangle Spline**

We can create another spline curve from 3 control points ABC in the following way: Point D is located proportion $t$ along AB. Point E is located proportion $t$ along BC. We take the locus of the intersection of AE and CD:

\[
\begin{align*}
X &= \frac{x_0 - 2tx_0 + tx_0^2 + tx_1 - tx_1^2 + tx_2}{1-t+t^2} \\
Y &= \frac{y_0 - 2ty_0 + ty_0^2 + ty_1 - ty_1^2 + ty_2}{1-t+t^2}
\end{align*}
\]

Copy the x coordinate into Maple and differentiate to get:

```maple
> u := diff((x[0]-2*x[0]*t+x[0]*t^2+x[1]*t-x[1]*t^2+x[2]*t^2)/(-t+1+t^2),t);
```

---

41
\[ u := \frac{-2x_0 + 2x_0 t + x_j - 2x_j t + 2x_2 t}{-t + 1 + t^2} \cdot \frac{(x_0 - 2x_0 t + x_0 t^2 + x_j t - x_j t^2 + x_2 t^2) (-1 + 2 t)}{(-t + 1 + t^2)^2} \]

Substituting \( t=0 \) and \( t=1 \):

\[
> \text{subs}(t=0,u);
\]  
\[-x_0 + x_j \]

\[
> \text{subs}(t=1,u);
\]  
\[-x_j + x_2 \]

Comparable result for \( y \) shows that the curve is tangent to the control triangle at the end points.
**Example 157: Another Triangle Spline**

We can also create a spline from a control triangle by taking the locus of a point G proportion \( t \) along DE.

Observing the parametric form of the curves we see that one is a parametric quadratic, while the other is a rational quadratic. Implicit forms are both conics (and almost, but not quite, identical).

\[
\begin{align*}
\text{Locus of G} \\
&\Rightarrow \begin{cases} 
X = 2\cdot b\cdot t + t^2 \cdot (a - 2\cdot b) \\
Y = 2\cdot c\cdot t \cdot (1 - t)
\end{cases} \\
&\Rightarrow 4\cdot Y \cdot a\cdot b \cdot c + 4\cdot X^2 \cdot c^2 - 4\cdot X \cdot a \cdot c^2 + Y^2 \cdot (a^2 - a \cdot b + b^2) + X \cdot Y \cdot (4 \cdot a \cdot c - 8 \cdot b \cdot c) = 0
\end{align*}
\]

\[
\begin{align*}
\text{Locus of F} \\
&\Rightarrow \begin{cases} 
X = \frac{t \cdot (b + a \cdot t - b \cdot t)}{1 - t + t^2} \\
Y = \frac{c \cdot (t - t^2)}{1 - t + t^2}
\end{cases} \\
&\Rightarrow Y \cdot a \cdot b \cdot c + X^2 \cdot c^2 - X \cdot a \cdot c^2 + Y^2 \cdot (a^2 - a \cdot b + b^2) + X \cdot Y \cdot (a \cdot c - 2 \cdot b \cdot c) = 0
\end{align*}
\]
What types of conics are they? Extending the curves a little can give a clue:

The blue curve looks like a parabola, the red certainly does not.

Copying the blue curve equation into Maple and examining the quadratic form shows that it is indeed a parabola:

\[
> 4c^2b^2aY + 4c^2X^2 - 4c^2aX + (a^2 - 4b^2a + 4b^2)Y^2 + (4ca - 8c^2b)XY = 0;
\]

\[
> <<4c^2 |(4c^2a - 8c^2b)/2>, <(4c^2a - 8c^2b)/2 | (a^2 - 4b^2a + 4b^2)Y^2 + (4ca - 8c^2b)XY = 0;
\]
How about the red curve:

\[
\begin{align*}
&c^2 b a Y + c^2 X^2 - c^2 a X + (a^2 - b a + b^2) Y^2 + (c a - 2 c b) Y X = 0
\end{align*}
\]

We see that the determinant is positive. This means we will always have a portion of an ellipse.