Optimal Placing of Crop Circles in a Rectangle

Abstract

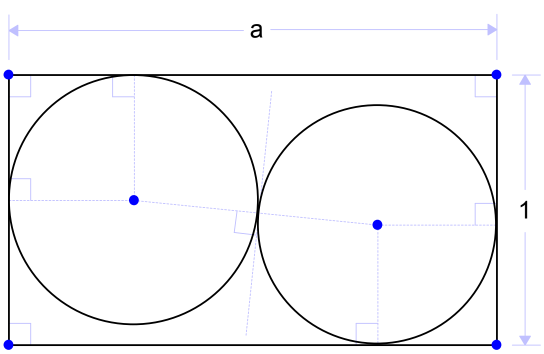
Many large-scale watering configurations for farming are done with circles because of the circle's practicality, but a circle obviously cannot tessellate a plane, nor do they fit very well in a rectangle. Unfortunately, most plots of land are rectangular. Additionally, there are occasionally extra obstacles, like roads or houses. In this paper, we use Geometry Expressions to investigate the optimal method of placing circles.

Problem

In this paper, we look at a simple situation: optimally placing two circles in a rectangle with aspect ratio less than 2. In addition to the size and location of the circles, there is another variable to consider. Instead of using a full circle, we can use a "pac-man" shape, or a sector. Thus, we also have to determine whether the larger or smaller circle should be the pac-man. We divide the investigation into two parts: one for aspect ratios between 1 and 1.5, and one for aspect ratios between 1.5 and 2. For simplicity, we assume the shorter side of the rectangle is 1, and the longer side will be *a*.

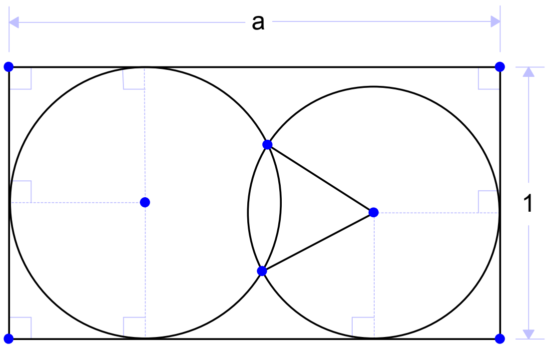
Solution

We consider a few cases . The simplest solution is to simply place two full circles, each tangent to two sides of the rectangle and the other circle (1.1).

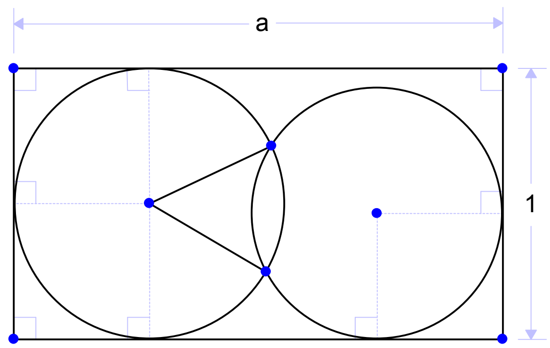


1.1

Or, we can have either the smaller circle (1.2) or the larger circle (1.3) be a pac-man.



[1.2]

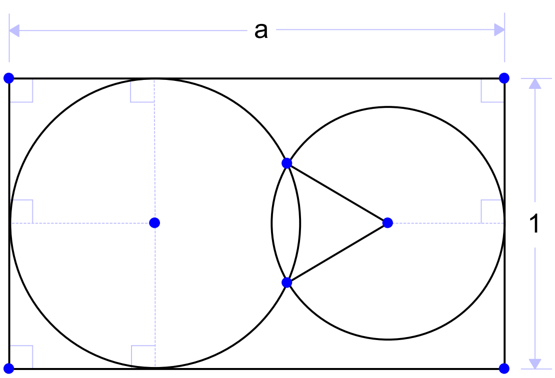


[1.3]

There are two variables we change, aspect ratio and radius, to alter the total area, so we would graph in 3D. However, due to the limitations of drawing a 3D graph on 2D paper, we simply take cross-sections of the 3D graph and look for the maximum area for the given aspect ratio.

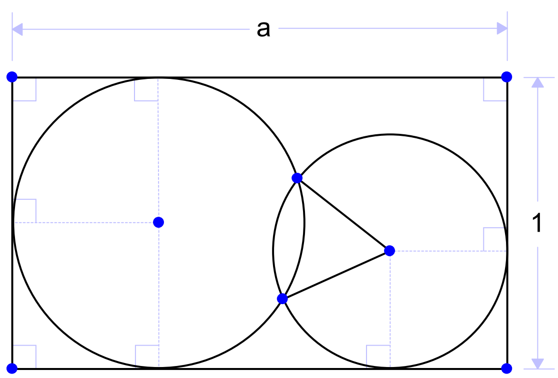
Lemma

First, we show that once the larger circle has been placed, it is always optimal to place the smaller circle in a corner, tangent to two sides of the rectangle, rather than centered, tangent to only one side of the rectangle (2.1).



[2.1]

VS



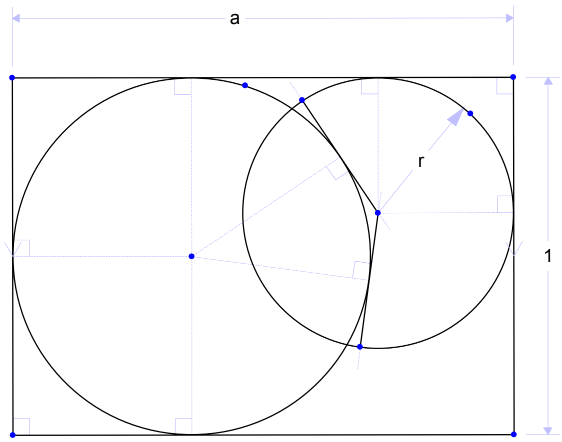
Take a full smaller circle with a given radius, the trivial case. The smaller circle is clearly the largest when it is in the corner, tangent to two sides. However, now consider the pac-man case. Take one circle with maximal radius, and take a given radius for the other circle. The total area is thus only dependent on the angle of the pac-man's "mouth." This angle will be at a minimum when the centers of the circles are furthest apart. Thus, the smaller circle should be in the corner, tangent to two sides and the other circle.

Part 1: Aspect Ratio between 1 and 1.5

Once we determine that the optimal placement of the smaller circle is in the corner and tangent to two sides, we determine if and when it is better to have the small circle as a pac-man (3.1) or the big circle as a pac-man (3.2).

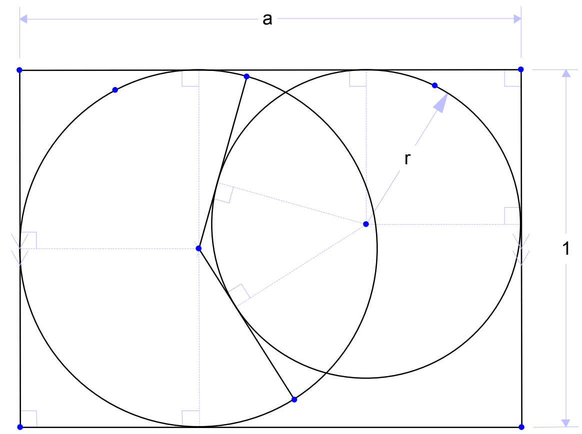
Using Geometry Expressions, we determine the angle of the fan in terms of aspect ratio (*a*) and radius (*r*). The two configurations are shown below with their corresponding angle formulae.

A: Small Circle as Pac-man



[3.1]

B: Big Circle as Pac-man



[3.2]

Angle Formulae:

$$\pi + \arctan\left(\frac{1}{2}\,\frac{\sqrt{1-4a+4a^2-8ra+8r^2}}{-a+a^2-2ra+2r^2}\right) \qquad \pi + \arctan\left(\frac{2\sqrt{1-2a+2a^2-4ra+2r^2}\,r\sqrt{2}}{1-2a+2a^2-4ra}\right)$$

With the angle formula the area covered by the circle and the pac-man as a percentage of total area possible can be determined.

Percentage of Area Covered (A):

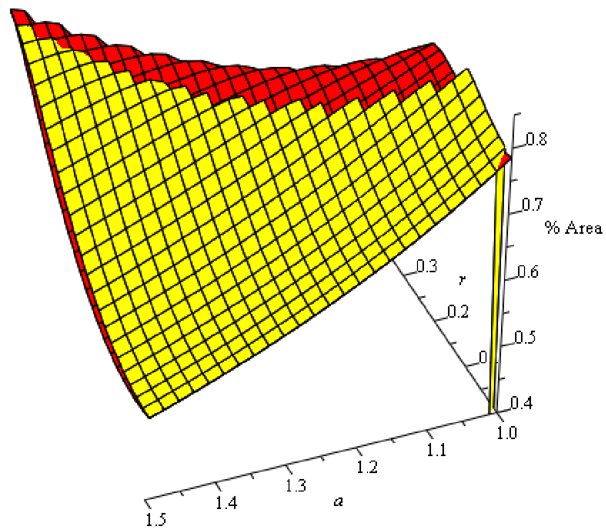
$$\frac{1}{2}\,\frac{.5\pi + r^2 - \arctan\left(\frac{1}{2}\,\frac{\sqrt{1-4a+4a^2-8ra+8r^2}}{-a+a^2-2ra+2r^2}\right)r^2}{a}$$

Percentage of Area Covered (B):

$$\begin{cases} \dfrac{.25\pi + r^2\pi - 0.125\arctan\left(\dfrac{2\sqrt{1-2a+2a^2-4ra+2r^2}\,r\sqrt{2}}{-a+a^2-2ra+2r^2}\right)}{a} : r < \dfrac{1}{4}\,\dfrac{1-2a+2a^2}{a} \\ \dfrac{.125\pi + r^2\pi - 0.125\arctan\left(\dfrac{2\sqrt{1-2a+2a^2-4ra+2r^2}\,r\sqrt{2}}{-a+a^2-2ra+2r^2}\right)}{a} : r > \dfrac{1}{4}\,\dfrac{1-2a+2a^2}{a} \end{cases}$$

Note: Because the inverse tangent has a limited range, the formula produces negative values for angles larger than $\frac{\pi}{2}$. Configuration A does not take into account the part of the equation for less than $\frac{\pi}{2}$ because angles greater than $\frac{\pi}{2}$ are clearly optimal. The piecewise function for Configuration B is split at *r* = $\frac{1}{4}\frac{1-2a+2a^2}{a}$, where is the angle is equal to $\frac{\pi}{2}$.

Percent area covered for configurations A and B are plotted in 3D below (3.3).

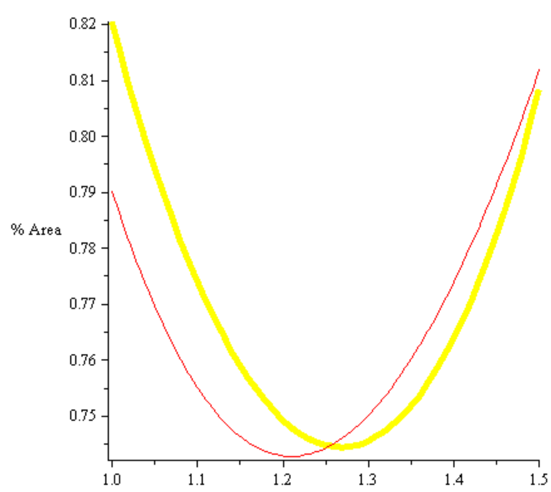


[3.3]

As seen in (3.1) and (3.2), there are two variables that determine the area covered by the circle and the pac-man. The graph above shows the percentage of area covered for all combinations of *a* (1.0-1.5) and *r* (0.0-0.5) for the given ranges. The yellow surface is A and the red surface is B. What we care about is the greatest percentage of area covered for each given aspect ratio from 1.0 to 1.5.

Using the Optimization program from Maple, we isolate *a* at intervals of 0.01 and find the maximum at each step for Configuration A and B. The outcome for the maximum radius and percent area covered is shown in the data table in the conclusion. The plot of the maximums at each interval of *a* is shown below (3.4). Once again, the yellow curve represents Configuration A and the red curve represents configuration B.

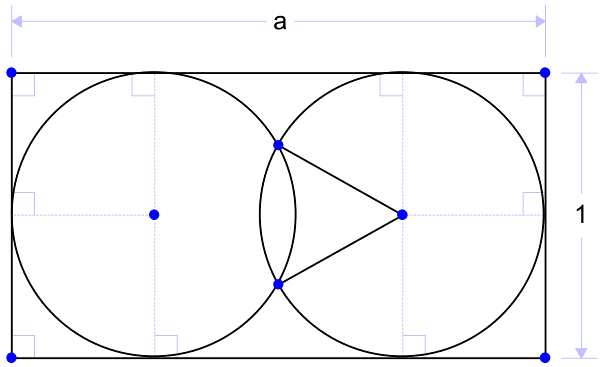
[3.4]



From comparing the lists produced by calculating the local maximum for every 0.01 interval of *a*, it is apparent that configuration A is better until the aspect ratio is approximately 1.26. At that point on until *a*=1.5, configuration B is optimal.

Part 2: Aspect Ratio between 1.5 and 2

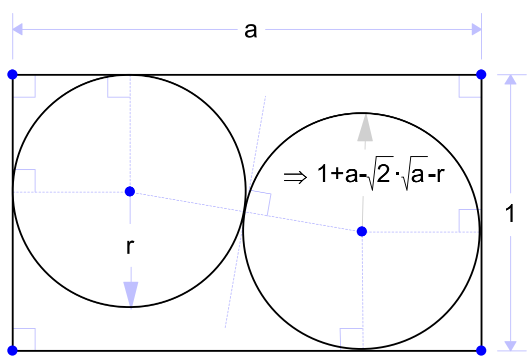
The obvious solution is to place two circles of radius .5, and make one a pac-man (4.1).



[4.1]

But is it the optimal configuration? Are there any cases where there is a better configuration? As it turns out, it is indeed the best configuration for most aspect ratios, but not all. We investigate some alternatives in more detail.

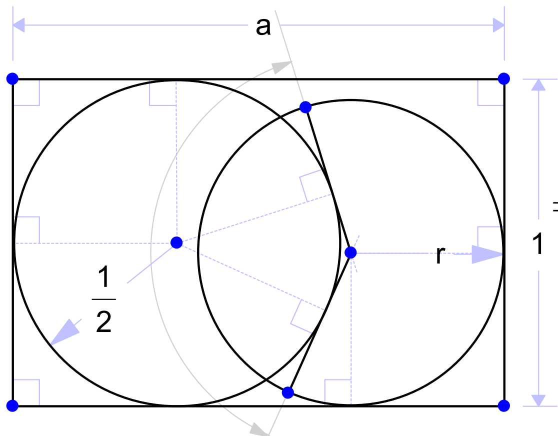
We look at three cases, as described in the solution outline. First, we examine the case of two full circles (4.2).



[4.2]

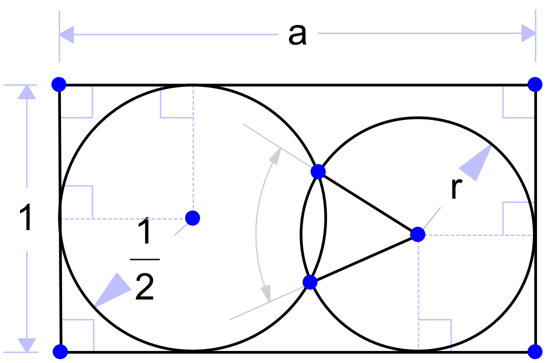
We denote the radius of one circle *r*. Geometry expressions then calculates the radius of the other circle as shown, and the total area of the two circles is $\pi\left(r^2+\left(1+a-\sqrt{2a}-r\right)^2\right)$. The total percent coverage is thus $\frac{\pi\left(r^2+\left(1+a-\sqrt{2a}-r\right)^2\right)}{a}$.

Next, we look at the two pac-man configurations, starting with the smaller circle as the pac-man. There are two cases: one with radii tangent to the larger circle (4.3) and one with radii endpoints on the larger circle (4.4). We now set the radius of the larger circle as $\frac{1}{2}$, and the radius of the smaller as *r*.



$$\Rightarrow \pi+\arctan\left(\frac{\sqrt{1-4\cdot a+4\cdot a^2-8\cdot a\cdot r+8\cdot r^2}}{2\cdot\left(-a+a^2-2\cdot a\cdot r+2\cdot r^2\right)}\right)$$

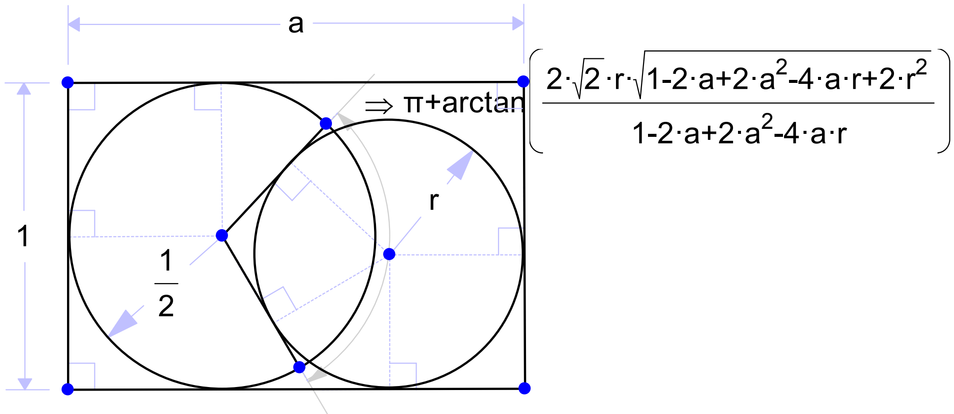
[4.3]



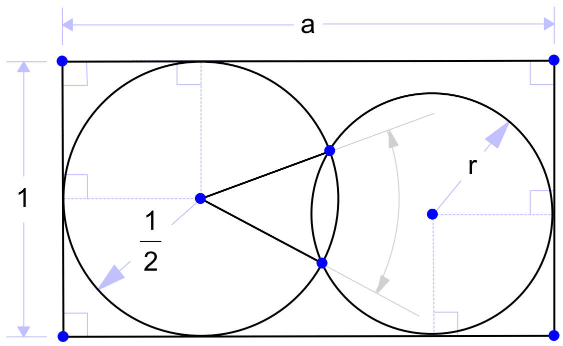
[4.4]

For (4.3), the percent coverage is $\frac{r^2\left(\pi-arctan\left(\frac{\sqrt{1-4a+4a^2-8ar+8r^2}}{2(-a+a^2-2ar+2r^2)}\right)\right)+\frac{\pi}{2}}{2a}$. Unfortunately, Geometry Expression's calculation for the angle in (4.4) is too large to fit on the page. However, we can still input the expression into Mathematica.

Now, we look at the bigger circle as the pac-man. Again, we have two cases where the radii are tangent to (4.5) and lying on (4.6) the smaller circle.



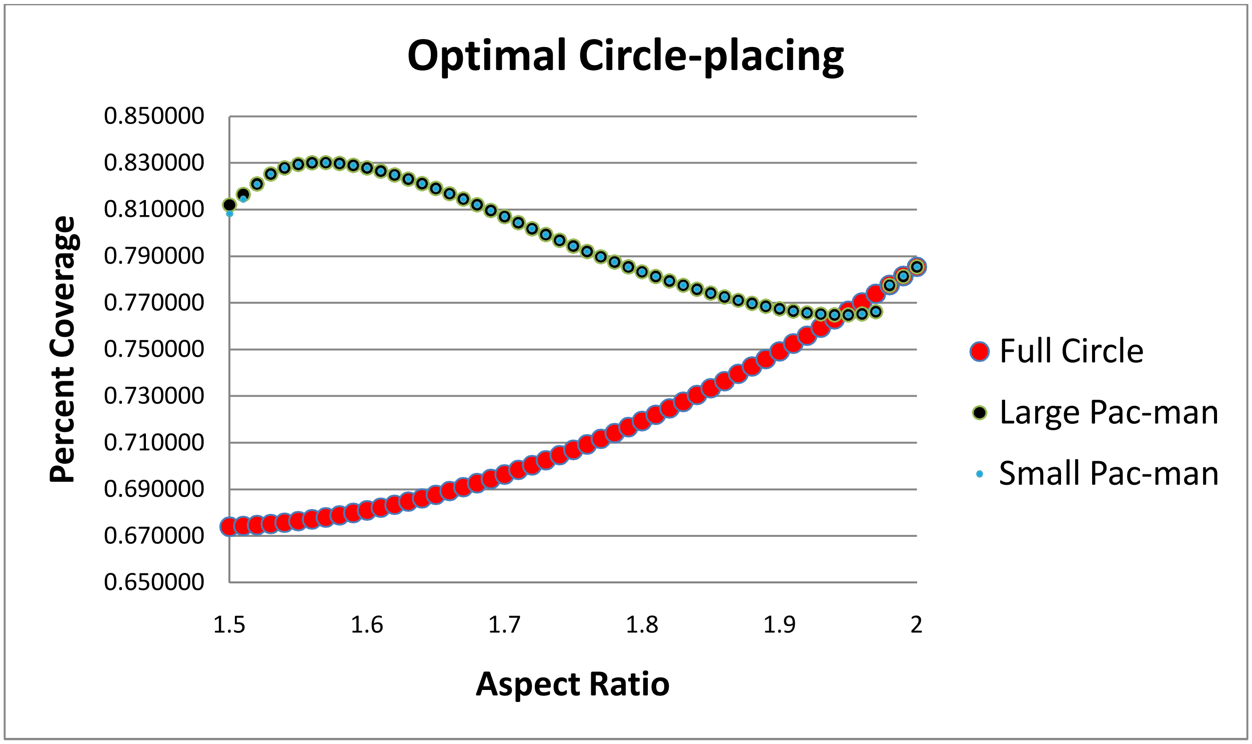
[4.5]



[4.6]

For (4.5), the percent coverage is $\frac{\pi - \arctan\left(\frac{2r\sqrt{2(1-2a+2a^2-4ar+2r^2)}}{1-2a+2a^2-4ar}\right) + 8\pi r^2}{8a}$. Again, the equation for the angle for (4.6) is unfortunately too big to fit on the page.

Now that we have the formulae for each case, we plot them to determine the optimal radii. Due to the impracticality of plotting a 3D graph as is necessary, we look at a set of given values for *a*, and determine the optimal *r* for each *a*. The results are summarized in this plot.



Conclusion

For practical purposes, we include a table with the appropriate data for aspect ratios to the nearest hundredth. The bolded column is the optimal strategy, with the ideal radius and percent coverage. Note that between ratios of 1.53 and 1.94, the optimal strategy is to simply have a circle and a pac-man, both with maximal radius. Thus, there is no "bigger" circle, so strategies A and B are essentially the same. At 1.95, it becomes preferable to have no pac-man at all, and two full circles instead.

	A-Small		B-Big		C-Full Circle	
Aspect ratio	% Covered	Radius	% Covered	Radius	% Covered	Radius
1	**0.820403**	**0.143927**	0.790225	0.223002	0.808517	0.085786
1.01	**0.814749**	**0.148804**	0.785581	0.228236	0.802112	0.088733
1.02	**0.809328**	**0.153745**	0.781199	0.233440	0.795905	0.091714
1.03	**0.804139**	**0.158752**	0.777077	0.238623	0.789893	0.094730
1.04	**0.799177**	**0.163824**	0.773209	0.243791	0.784071	0.097779
1.05	**0.794441**	**0.168962**	0.769591	0.248950	0.778436	0.100862
1.06	**0.789927**	**0.174165**	0.766219	0.254103	0.772984	0.103978
1.07	**0.785635**	**0.179433**	0.763090	0.259254	0.767711	0.107126
1.08	**0.781562**	**0.184768**	0.760200	0.264406	0.762614	0.110306
1.09	**0.777706**	**0.190170**	0.757545	0.269561	0.757689	0.113518
1.1	**0.774065**	**0.195638**	0.755121	0.274720	0.752933	0.116760
1.11	**0.770638**	**0.201173**	0.752925	0.279886	0.748344	0.120034
1.12	**0.767424**	**0.206777**	0.750954	0.285058	0.743917	0.123337
1.13	**0.764422**	**0.212448**	0.749204	0.290238	0.739651	0.126670

1.14	**0.761630**	**0.218189**	0.747672	0.295426	0.735542	0.130033
1.15	**0.759048**	**0.223999**	0.746355	0.300622	0.731587	0.133425
1.16	**0.756675**	**0.229879**	0.745249	0.305827	0.727784	0.136845
1.17	**0.754510**	**0.235831**	0.744352	0.311041	0.724130	0.140294
1.18	**0.752554**	**0.241854**	0.743659	0.316264	0.720622	0.143771
1.19	**0.750805**	**0.247950**	0.743169	0.321495	0.717259	0.147275
1.2	**0.749265**	**0.254120**	0.742876	0.326734	0.714038	0.150807
1.21	**0.749265**	**0.260365**	0.742780	0.331982	0.710956	0.154365
1.22	**0.746808**	**0.266686**	0.742876	0.337238	0.708012	0.157950
1.23	**0.745893**	**0.273084**	0.743162	0.342502	0.705203	0.161561
1.24	**0.745187**	**0.279561**	0.743634	0.347773	0.702527	0.165198
1.25	**0.744692**	**0.286118**	0.744289	0.353051	0.699982	0.168861
1.26	0.744408	0.292756	**0.745124**	**0.358337**	0.697566	0.172549
1.27	0.744336	0.299476	**0.746137**	**0.363629**	0.695277	0.176262
1.28	0.744478	0.306282	**0.747324**	**0.368928**	0.693113	0.180000
1.29	0.744836	0.313174	**0.748683**	**0.374233**	0.691073	0.183762
1.3	0.745412	0.320155	**0.750210**	**0.379545**	0.689155	0.187548
1.31	0.745412	0.327226	**0.751903**	**0.384863**	0.687356	0.191359
1.32	0.747223	0.334390	**0.753759**	**0.390187**	0.685676	0.195192
1.33	0.748465	0.341649	**0.755775**	**0.395517**	0.684112	0.199049
1.34	0.749933	0.349006	**0.757948**	**0.395517**	0.682664	0.202929
1.35	0.751632	0.356463	**0.760275**	**0.406194**	0.681329	0.206832
1.36	0.753565	0.364024	**0.762754**	**0.411542**	0.680105	0.210758
1.37	0.755735	0.371692	**0.765382**	**0.416896**	0.678993	0.214705
1.38	0.758148	0.379470	**0.768156**	**0.422256**	0.677989	0.218675
1.39	0.760807	0.387362	**0.771075**	**0.427622**	0.677093	0.222667
1.4	0.763718	0.395372	**0.774134**	**0.432994**	0.676303	0.226680
1.41	0.766886	0.403505	**0.777333**	**0.438373**	0.675618	0.230714
1.42	0.770317	0.411765	**0.780668**	**0.443758**	0.675037	0.234770
1.43	0.774018	0.420157	**0.784136**	**0.449150**	0.674558	0.238847
1.44	0.777997	0.428688	**0.787737**	**0.454549**	0.674180	0.242944
1.45	0.782261	0.428688	**0.791466**	**0.459955**	0.673902	0.247061
1.46	0.786820	0.446191	**0.795323**	**0.465368**	0.673723	0.251199
1.47	0.791682	0.455178	**0.799304**	**0.470788**	0.673641	0.255357
1.48	0.796860	0.464332	**0.803408**	**0.476216**	0.673655	0.259535
1.49	0.802365	0.473664	**0.807633**	**0.481653**	0.673765	0.263732
1.5	0.808211	0.483184	**0.811976**	**0.487097**	0.673969	0.267949
1.51	0.814413	0.492904	**0.816436**	**0.492549**	0.674266	0.272185
1.52	0.820828	0.500000	**0.821010**	**0.498010**	0.674655	0.276440
1.53	**0.825239**	**0.500000**	**0.825239**	**0.500000**	0.675135	0.280714
1.54	**0.827874**	**0.500000**	**0.827874**	**0.500000**	0.675705	0.285007
1.55	**0.829369**	**0.500000**	**0.829369**	**0.500000**	0.676364	0.289318
1.56	**0.830054**	**0.500000**	**0.830054**	**0.500000**	0.677111	0.293648
1.57	**0.830128**	**0.500000**	**0.830128**	**0.500000**	0.677946	0.297995

1.58	**0.829723**	**0.500000**	**0.829723**	**0.500000**	0.678866	0.302361
1.59	**0.828932**	**0.500000**	**0.828932**	**0.500000**	0.679872	0.306745
1.6	**0.827824**	**0.500000**	**0.827824**	**0.500000**	0.680962	0.311146
1.61	**0.826451**	**0.500000**	**0.826451**	**0.500000**	0.682136	0.315564
1.62	**0.824855**	**0.500000**	**0.824855**	**0.500000**	0.683393	0.320000
1.63	**0.823067**	**0.500000**	**0.823067**	**0.500000**	0.684731	0.324453
1.64	**0.821116**	**0.500000**	**0.821116**	**0.500000**	0.686151	0.328923
1.65	**0.819023**	**0.500000**	**0.819023**	**0.500000**	0.687650	0.333410
1.66	**0.816806**	**0.500000**	**0.816806**	**0.500000**	0.689230	0.337913
1.67	**0.814483**	**0.500000**	**0.814483**	**0.500000**	0.690888	0.342433
1.68	**0.812066**	**0.500000**	**0.812066**	**0.500000**	0.692623	0.346970
1.69	**0.809568**	**0.500000**	**0.809568**	**0.500000**	0.694436	0.351522
1.7	**0.806997**	**0.500000**	**0.806997**	**0.500000**	0.696326	0.356091
1.71	**0.804369**	**0.500000**	**0.804369**	**0.500000**	0.698291	0.360676
1.72	**0.801772**	**0.500000**	**0.801772**	**0.500000**	0.700331	0.365276
1.73	**0.799236**	**0.500000**	**0.799236**	**0.500000**	0.702446	0.369892
1.74	**0.796761**	**0.500000**	**0.796761**	**0.500000**	0.704634	0.374524
1.75	**0.794350**	**0.500000**	**0.794350**	**0.500000**	0.706895	0.379171
1.76	**0.792003**	**0.500000**	**0.792003**	**0.500000**	0.709229	0.383834
1.77	**0.789722**	**0.500000**	**0.789722**	**0.500000**	0.711634	0.388511
1.78	**0.787508**	**0.500000**	**0.787508**	**0.500000**	0.714110	0.393204
1.79	**0.785363**	**0.500000**	**0.785363**	**0.500000**	0.716657	0.397911
1.8	**0.783289**	**0.500000**	**0.783289**	**0.500000**	0.719273	0.402633
1.81	**0.781290**	**0.500000**	**0.781290**	**0.500000**	0.721959	0.407370
1.82	**0.779368**	**0.500000**	**0.779368**	**0.500000**	0.724713	0.412122
1.83	**0.777527**	**0.500000**	**0.777527**	**0.500000**	0.727536	0.416887
1.84	**0.775771**	**0.500000**	**0.775771**	**0.500000**	0.730425	0.421667
1.85	**0.774105**	**0.500000**	**0.774105**	**0.500000**	0.733382	0.426462
1.86	**0.772535**	**0.500000**	**0.772535**	**0.500000**	0.736405	0.431270
1.87	**0.771068**	**0.500000**	**0.771068**	**0.500000**	0.739494	0.436092
1.88	**0.769714**	**0.500000**	**0.769714**	**0.500000**	0.742648	0.440928
1.89	**0.768483**	**0.500000**	**0.768483**	**0.500000**	0.745866	0.445778
1.9	**0.767389**	**0.500000**	**0.767389**	**0.500000**	0.749149	0.450641
1.91	**0.766449**	**0.500000**	**0.766449**	**0.500000**	0.752496	0.455518
1.92	**0.765686**	**0.500000**	**0.765686**	**0.500000**	0.755905	0.460408
1.93	**0.765130**	**0.500000**	**0.765130**	**0.500000**	0.759378	0.465312
1.94	**0.764822**	**0.500000**	**0.764822**	**0.500000**	0.762912	0.470228
1.95	0.764824	0.500000	0.764824	0.500000	**0.766508**	**0.475158**
1.96	0.765228	0.500000	0.765228	0.500000	**0.770166**	**0.480101**
1.97	0.766195	0.500000	0.766195	0.500000	**0.773884**	**0.485057**
1.98	0.777659	0.490025	0.777428	0.490025	**0.777662**	**0.490025**
1.99	0.781496	0.495006	0.781287	0.495006	**0.781500**	**0.495006**
2	0.785397	0.500000	0.785397	0.500000	**0.785397**	**0.500000**

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