# Stained Glass Design

**Teaching Goals:**

1. Students apply graphic methods to design various shapes on the plane.
2. Students apply geometric transformations of graphs of functions in order to design various shapes on the plane.
3. Students find equations of all curves obtained by geometric transformations in general form in either Cartesian or parametric form as defined by the teacher.
4. Students will determine domains of the independent variable for the equations of all curves for the design of the shape they construct in either Cartesian or parametric form as defined by the teacher.
5. Students will verify these results with the help of the software.

**Prior Knowledge**

- Students should know graphs of linear functions and the equation of a circle in Cartesian and parametric forms.

- Students should know the coordinate form of a translation by a vector along each coordinate axis.

- Students should know that the equation $y = f(x - a)$ corresponds to the translation of the curve with the equation $y = f(x)$ by vector $(a, 0)$ along the *x-axis.*

- Students should know that equation $y = f(x) + b$ corresponds to the translation of the curve with the equation $y = f(x)$ by vector $(0, b)$ along the *y-axis.*

- Students should know that a counter clockwise rotation has a positive angle and a clockwise rotation has negative angle, with angles in the range $[0, \pi]$.

- Students should know that the equation $y = -f(x)$ corresponds to the reflection of the curve with the equation $y = f(x)$ about the *x-axis.*

- Students should know that equation $y = f(-x)$ corresponds to the reflection of the curve with the equation $y = f(x)$ about the *y-axis.*

**Problem:**

Construct a stained glass design as shown in the picture below. This design is created by using the curves given by the following equations: $y = |x|$ on [-3, 3] and $x^2 + y^2 = a^2$ on $[0, |a|]$. Try to recreate a similar design and to create other designs that could be created by these curves.
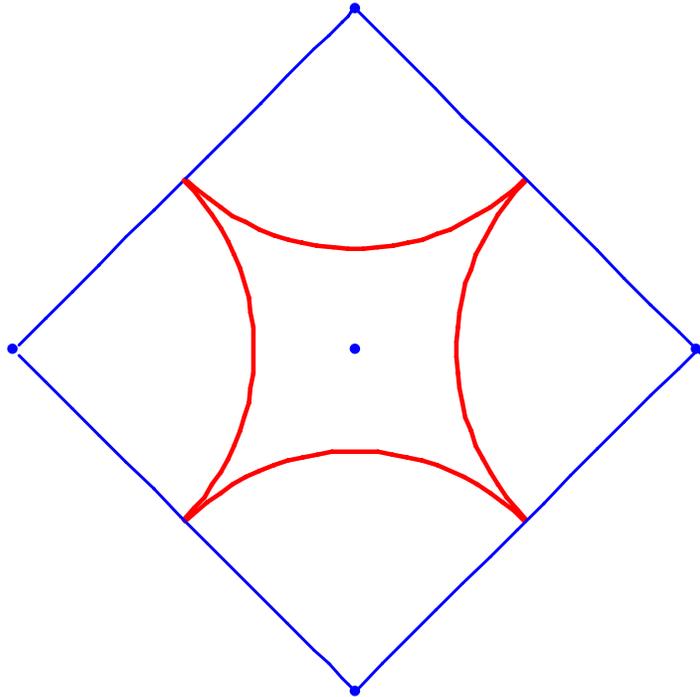
Determine equations of all curves on your design with their domain and verify them with the help of the software.

**Comment:** the idea of this problem is to be able to send this information to another person who should be able to recreate this design based on the information provided (not by mindless copying).

### *Part 1 – Setting Up Problem in Geometry Expressions*

1.  Open a new file. If axes do not appear in the blank document, create them by clicking the **Toggle grid and axes** icon on the top toolbar.

2.  Select **Function** from the **Draw** toolbox.  In the Type: box click the down arrow on the right and select Parametric from the list. In the next row, X= , enter: *T* ; in the next row, Y=, enter: *abs(T)*; for Start: enter: *-3*; for End: enter: *3.*

3.  Click the equation of the function,  right-click and select **Hide** from the context menu. Select the curve, right-click and select **Properties** from the context menu.  Click the Line Style row and it's down arrow to change it to Dot.

4.  Select **Function** from the **Draw** toolbox.  In the Type: box click the down arrow on the right and select Parametric from the list. In the next row, X= , enter: *a\*cos(T)* ; in the next row, Y=, enter: *a\*sin(T)*; for Start: enter: *0*; for End: enter: *1.57.*

5.  Click the equation of the function, right-click and select **Hide** from the context menu.  Select

the curve, right-click and select **Properties** from the context menu.  Click the Line Style row and it's down arrow to change it to Dot.
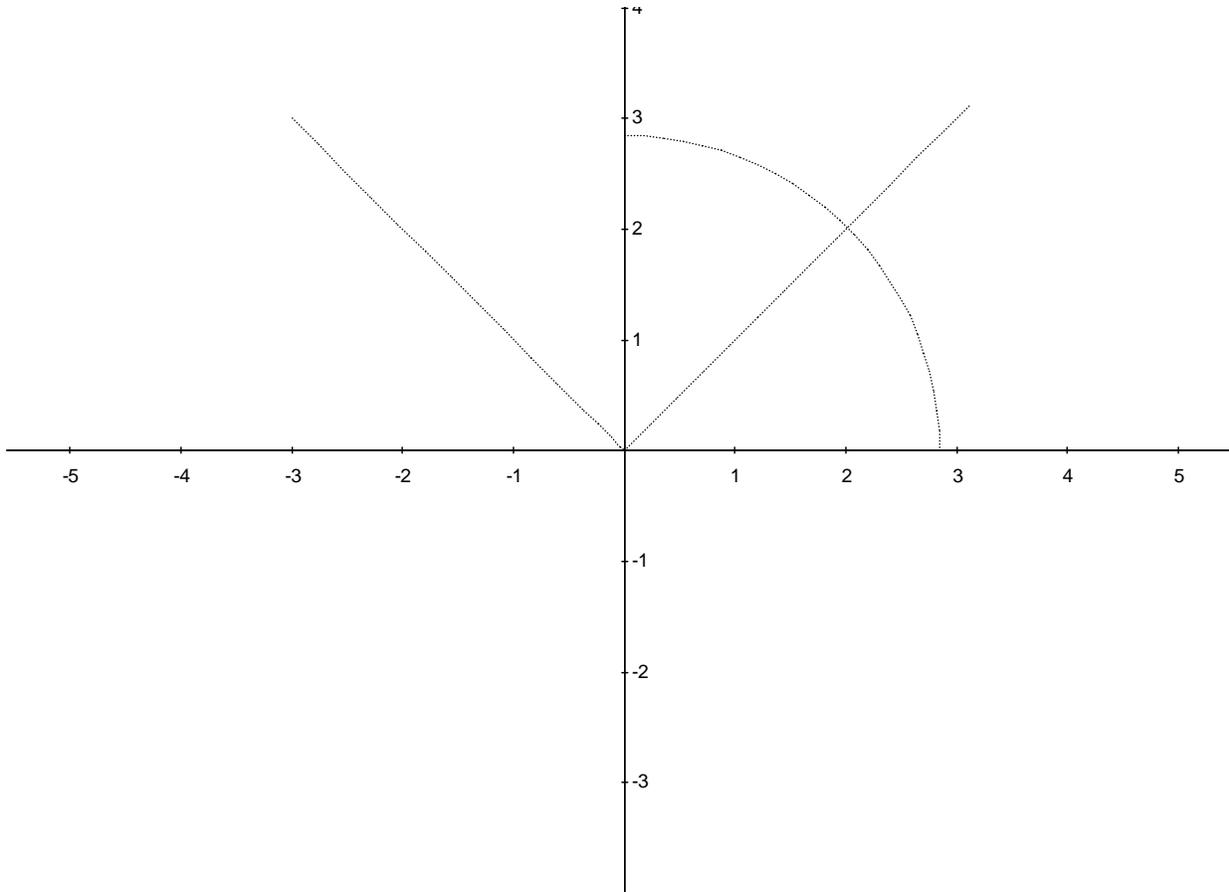


### *Part 2 – Creating Stained Glass Design*

**Q1**: How can you construct a square from the given curves?

**A**: It is expected that students will use translation along y-axis and reflection about x-axis according to the following steps, for example:

1.  Click the straight line function and select **Translation** from the **Construct** toolbox. Construct a vector of translation from the origin down along the y-axis. The translated function will appear on the screen.

2.  Select the end point of the translation vector and click **Coordinate** in the **Constrain** toolbox. Enter (0,-3) for the coordinates of the endpoint of the translation vector.

3.  Select the translation vector, right-click and select **Hide** from the context menu.

4.  Select the translated function. Right-click and select **Properties** from the context menu. Change the Line Color to Blue and the Line Style to Solid 3.

5. Click the translated function and select **Reflection** from the **Construct** toolbox. Click the x-axis and the function will be reflected over the x-axis.  Adjust the line color and style as in step 4 above.
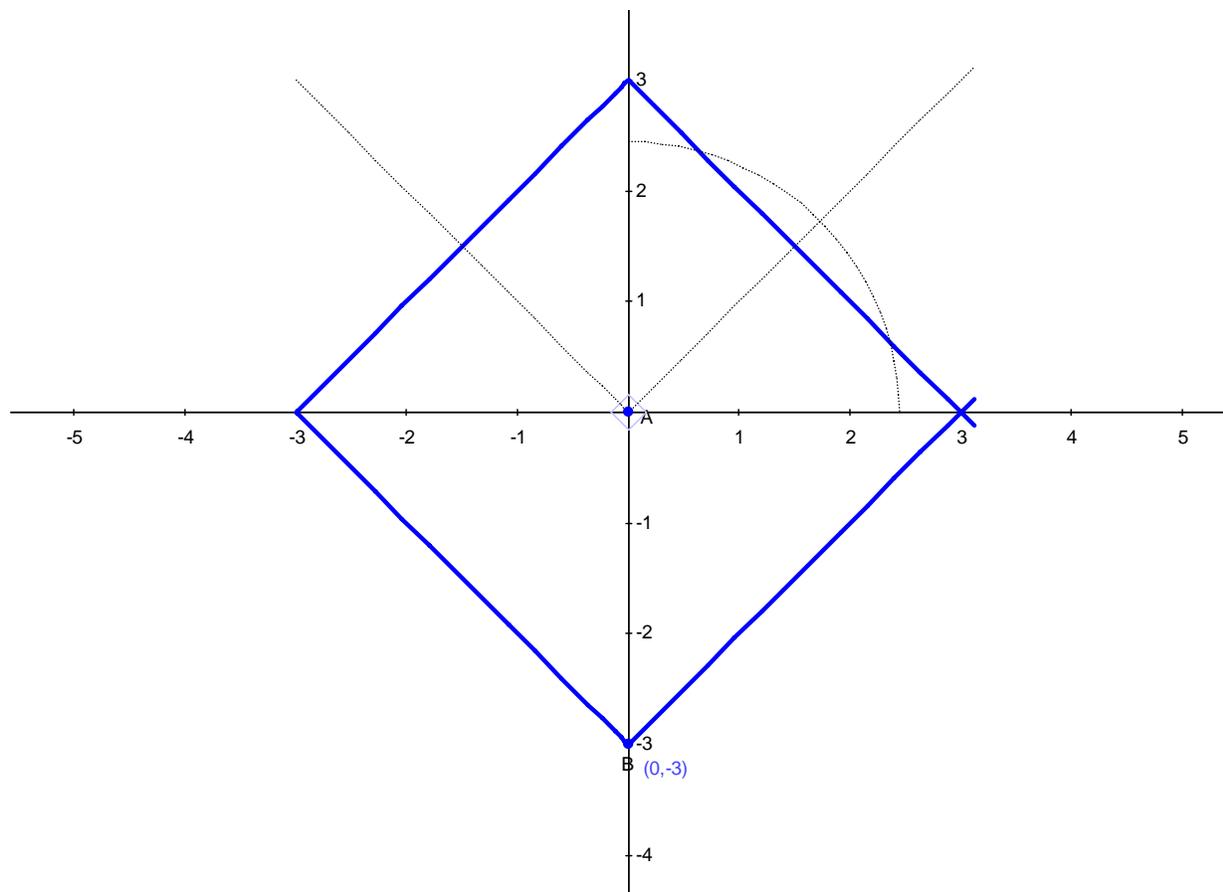


**Q2**: How can you construct a quarter circle with its center at the bottom vertex of the square?

**A**: Students may suggest rotating the given quarter circle by $\frac{\pi}{4}$ about the origin and then translating the center of the circle down to the point (-3, 0). Here are the steps:

1. Select the arc and click **Rotation** from the **Construct** toolbox. Click the origin and the box will appear for entering the angle. (If the **Symbols** toolbox is not displayed, click **View / Tool Panels / Symbols**)  $\pi$ can be found in the lower right corned of the dialog.  Enter *π/4.*

2. Select the angle of rotation label and **Hide** it.

3. Select the rotated arc and click **Translation** from the **Construct** toolbox. Construct the translation vector from the origin to the point (0,-3). **Hide** the intermediate copy of the arc and the translation vector.

4. Change the new arc's line color to Red and the line style to Solid 3.

**Q3.** How can you construct a quarter circle with the center in the top vertex of the square?

**A**: Students may suggest similar steps as above, but the simplest way to do this is to reflect the newly constructed arc about the x-axis. The following steps will accomplish this:

1. Click the arc with its center at the bottom vertex of the square.  Select **Rotation** from the **Construct** toolbox and select the x-axis. The image will appear on the screen.

2. Select the arc and right-click **Properties** to change the line color to Red and the line style to Solid 3.
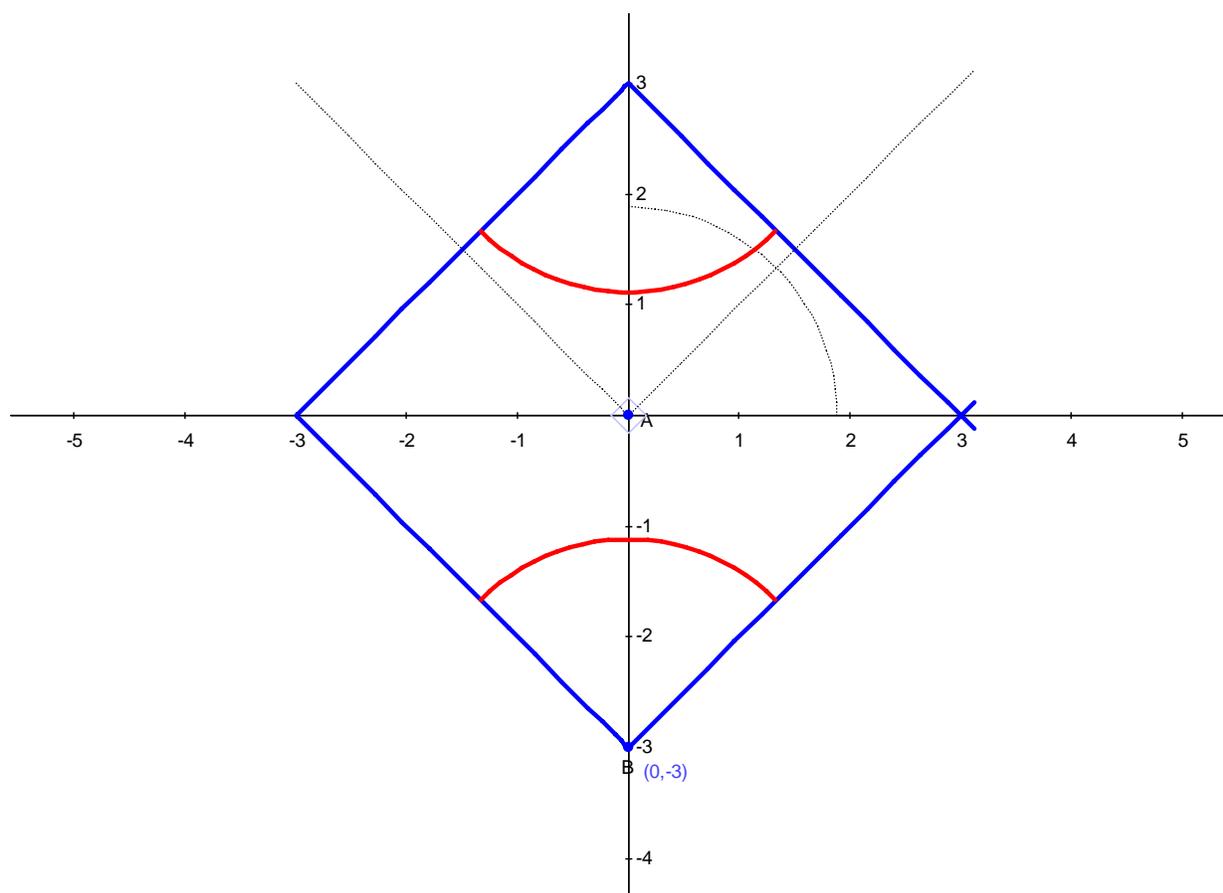


**Q4**: How can you construct a quarter circle with the center in the left vertex of the square?

**A**: Students may suggest rotating the given quarter circle by -π/4 about the origin and then translating the center of the circle down to the point (-3, 0). Here are the steps:

1. Select the original arc and click **Rotation** from the **Construct** toolbox. Click the origin and the angle edit box will appear. From the **Symbols** toolbox, find π and type: *-π/4.*

2. Select the angle of rotation label and **Hide** it.

3. Select the rotated arc and click **Translation** from the **Construct** toolbox. Construct the translation vector from the origin to the point (-3, 0). **Hide** the intermediate copy of the arc and the translation vector.

4. Select the arc and right-click **Properties** to change the line color to Red and the line style

to Solid 3.

**Q5.** How can you construct a quarter circle with the center in the right vertex of the square?

**A**: Students may suggest similar steps as above, but the simplest way to do this is to reflect the arc with the center at the left vertex about the y-axis. The following steps will accomplish this:

1.  Click the arc and select **Reflection** from the **Construct** toolbox. Click the y-axis and the image will appear on the screen.

2.  Select the image and right-click **Properties** to change the line color to Red and the line style to Solid 3.

3.  Click-and-drag the original quarter circle arc to adjust the transformed arcs to the desired size.
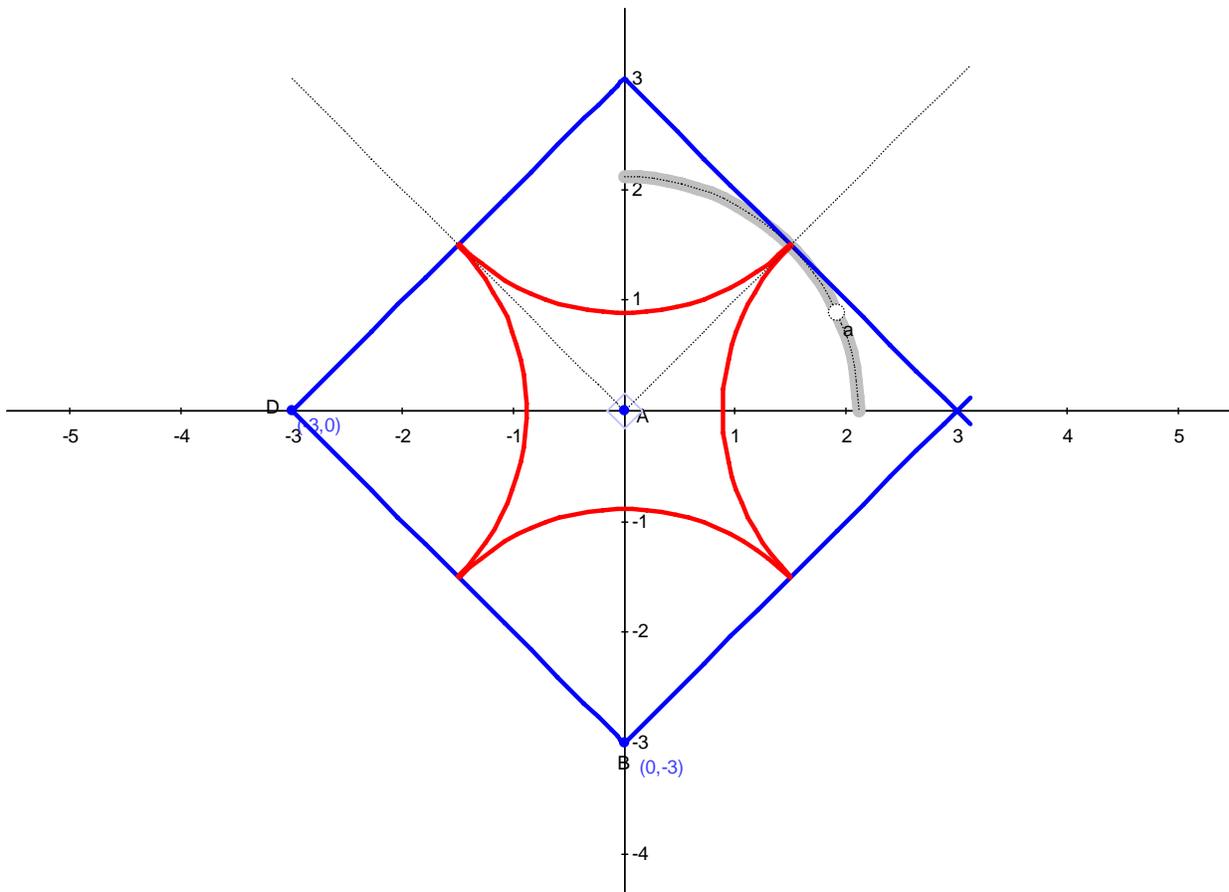


## Part 3 – Finding Equations of the Curves in the Stained Glass Design

In this part of the problem students have to determine the equation of each curve or function produced by transformation of the original curves. Based on the students' prior knowledge, the teacher can determine in which form students will write the equations of the curves: Cartesian or parametric. Students should also determine the domain of the independent variable for the

equations of each curve. This should be completed independently and then verified with the help of the software.

**A.** A Square is formed by two curves: $y = |x| - 3$, $[-3,3]$ produced by the equation $y = |x|$, $[-3,3]$ translated by vector (0, - 3) and $y = -|x| + 3$, $[-3,3]$ produced by equation $y = |x|$, $[-3,3]$ reflected about x-axis and translated by vector (0, 3). The inner design is made of four quarter circles. When the circles are tangent, the radius *a* of the circle is equal to half of the side of the square, $a = \dfrac{3\sqrt{2}}{2}$, and the x-coordinates of the points where the arcs intersect the squares are -1.5 and 1.5. The student can create the equations of the arcs as bounded circles centered at the origin, translated to the vertices of the square. Thus, the equations of the arcs of the circles follow.

At the bottom vertex (0,-3): the circle $x^2 + y^2 = \dfrac{9}{2}$ , translated by vector (0,-3) becomes $x^2 + (y+3)^2 = \dfrac{9}{2}$, on [-1.5, 1.5] ; at the top vertex (0, 3): the bottom arc reflected about x-axis becomes $x^2 + (y-3)^2 = \dfrac{9}{2}$ on [-1.5, 1.5]; at the left vertex (-3, 0): the circle $x^2 + y^2 = \dfrac{9}{2}$ , translated by vector (-3,0) becomes $(x+3)^2 + y^2 = \dfrac{9}{2}$, on $[-1.5, -3+1.5\sqrt{2}]$; and at the right vertex (3, 0): the left arc reflected about y-axis becomes $(x-3)^2 + y^2 = \dfrac{9}{2}$, on $[3-1.5\sqrt{2}, 1.5]$. Students may have difficulty finding the domains of the functions.

### *Part 4 – Verification of the Equations with Geometry Expressions*

Here are the steps to find the equations using *Geometry Expressions*:

1. Select each curve, one at a time, and choose **Implicit Equation** from the **Calculate** toolbox. The equation of the curve will appear on the screen.

Figure showing:
- $\Rightarrow 9+X^2-6 \cdot Y+Y^2-a^2=0$
- $\Rightarrow Y=-\left(-3+|X|\right)$
- $\Rightarrow 9+6 \cdot X+X^2+Y^2-a^2=0$
- $\Rightarrow 9-6 \cdot X+X^2+Y^2-a^2=0$
- $\Rightarrow Y=-3+|X|$
- $\Rightarrow 9+X^2+6 \cdot Y+Y^2-a^2=0$

Points: D (−3,0), A, B (0,−3)
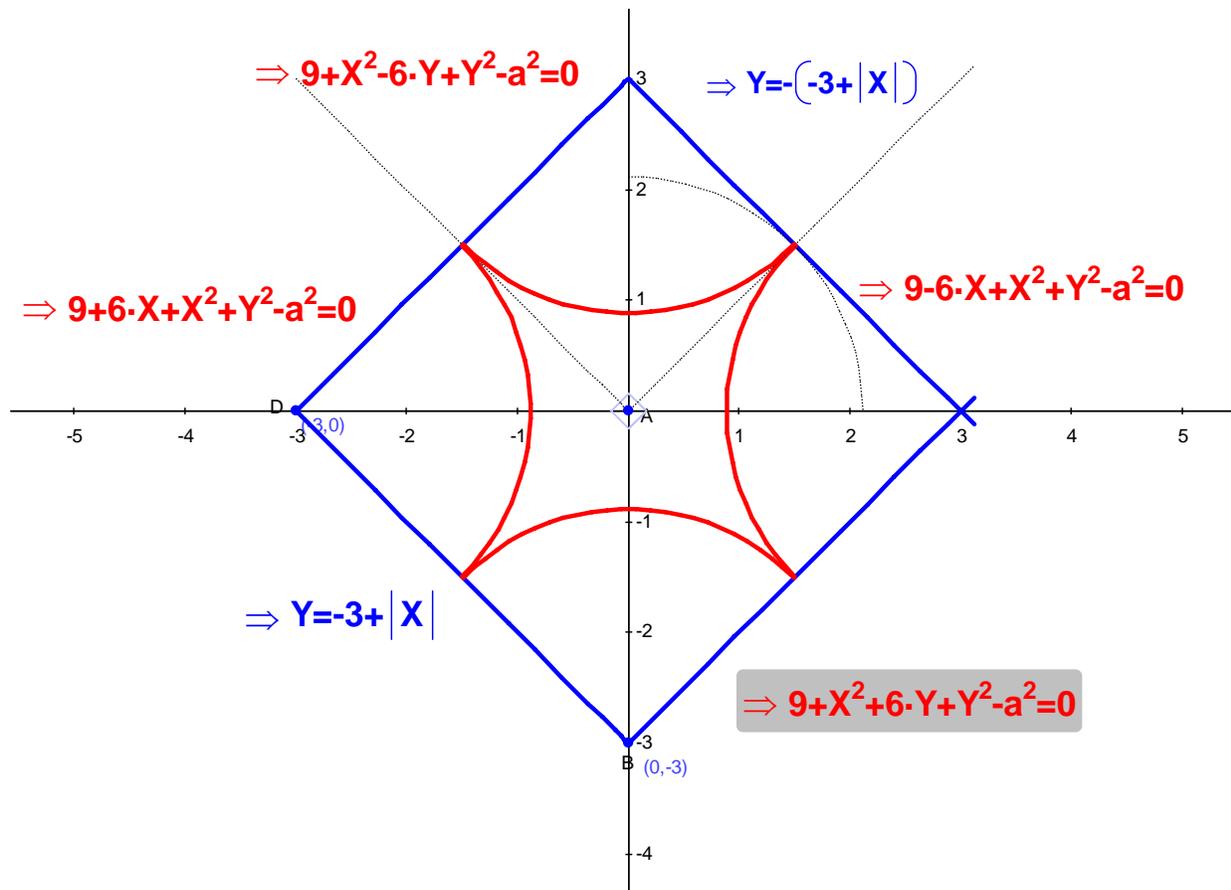
Students should substitute the value of *a* and verify that the equations given by the software are the same as ones they derived. In order to verify points of intersections, students can construct a square and a circle in a new document and determine points of intersection using the following steps:

1. Open a new file. If axes do not appear in the blank document, create them by clicking the **Toggle grid and axes** icon on the top toolbar.

2. Select **Point** from the **Draw** toolbox and construct four points for the vertices of the square.

3. Select one of the points, choose **Coordinate** from the **Constrain** toolbox and type (0, 3). Constrain the other points with the coordinates (0, -3), (3, 0) and (-3, 0).

4. Select **Line Segment** from the **Draw** toolbox and connect the points to construct a square.

5. Select **Circle** from the **Draw** toolbox and construct a circle with the center at the point (3, 0). With the circle selected choose **Radius** from the **Constrain** toolbox and enter *a* for the value.

6. Holding the *CTRL* key, select the circle and a segment in the 1st quadrant. Choose **Intersection** from the **Construct** toolbox. The point of intersection will appear.

7. Select the intersection point, choose **Coordinate** from the **Calculate (Symbolic)** toolbox and the coordinates of the intersection point will be displayed.

$$\Rightarrow \left( 3 - \frac{\sqrt{2}\cdot a}{2}, \frac{\sqrt{2}\cdot a}{2} \right)$$
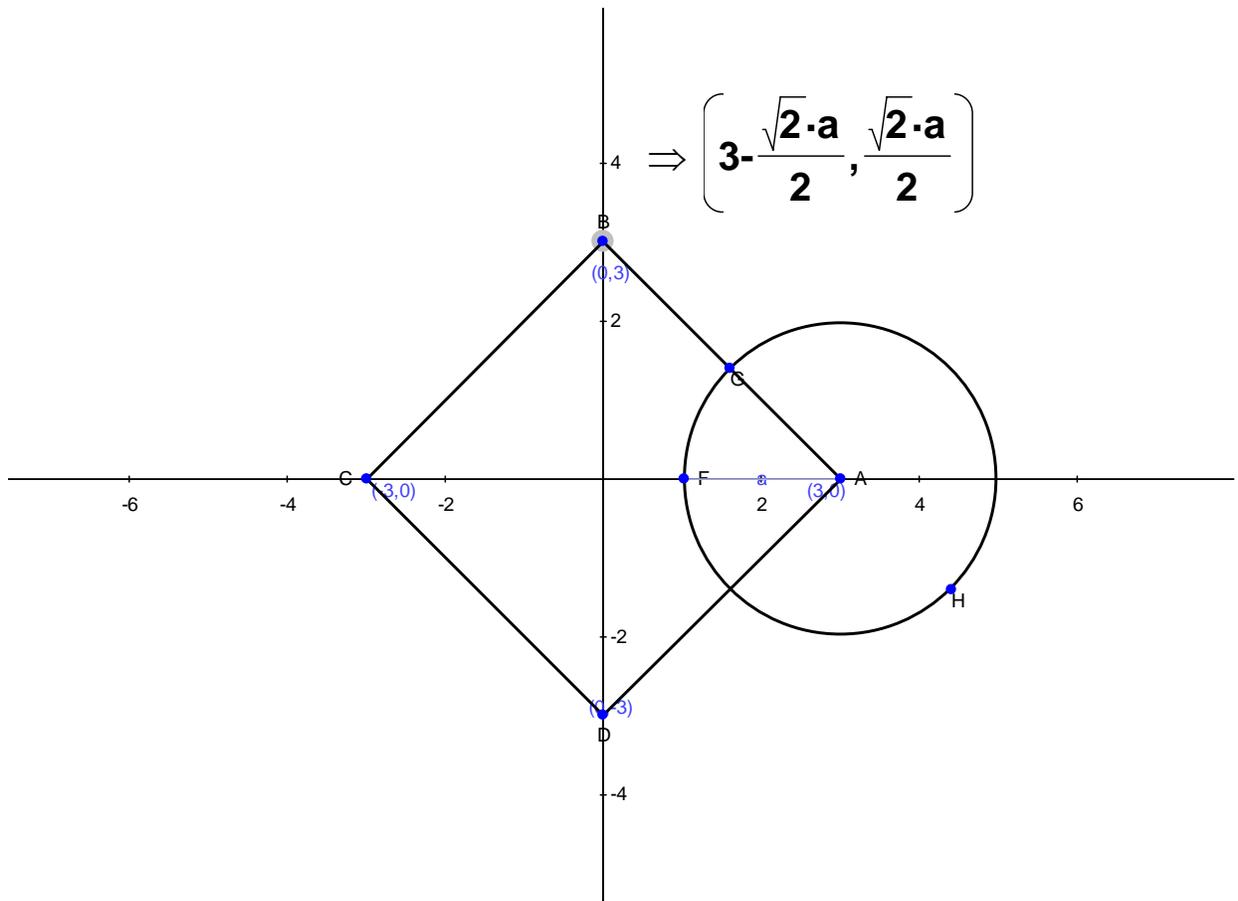
**Comment:** students can substitute the value of *a* and find the other intersection points based on symmetry.

### *Extensions*

1.  Students can consider the following cases for the stained glass designs

    a. when $a < \dfrac{3\sqrt{2}}{2}$ (arcs do not intersect)

    b. when $\dfrac{3\sqrt{2}}{2} < a < 3$ (each pair of arcs intersect)

    c. when $a > 3$

2.  Investigate case b where each pair of arcs intersects. Try to make the sum of the areas of the quadrilateral formed outside the arcs and the regions formed by the overlapping sectors in a given proportion, for example, 1:4 to the total area of the square. This problem should be solved by approximation, using a square to approximate the shape formed in the center outside the arcs.
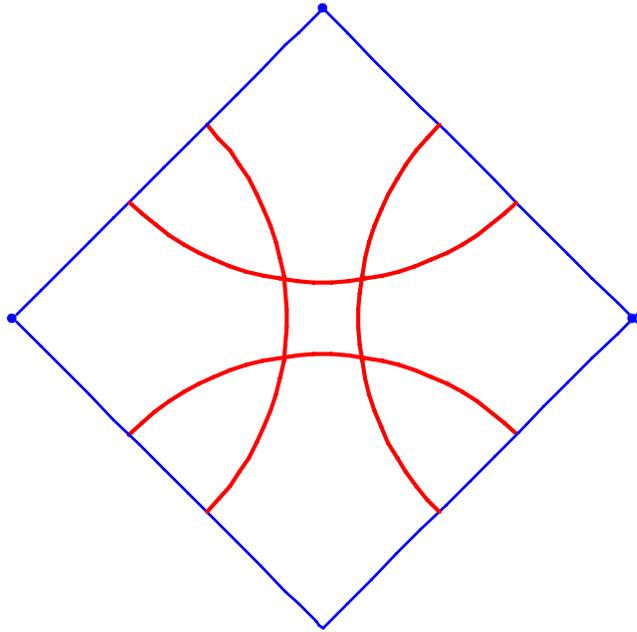
Students can use circles and segments and construct part of the design in a new file and then find coordinates of the intersection point for the vertex of the central "square" in terms of parameter *a*.



$$\Rightarrow \left[ \frac{3}{2} - \frac{\sqrt{-9+2 \cdot a^2}}{2}, \frac{3}{2} - \frac{\sqrt{-9+2 \cdot a^2}}{2} \right]$$
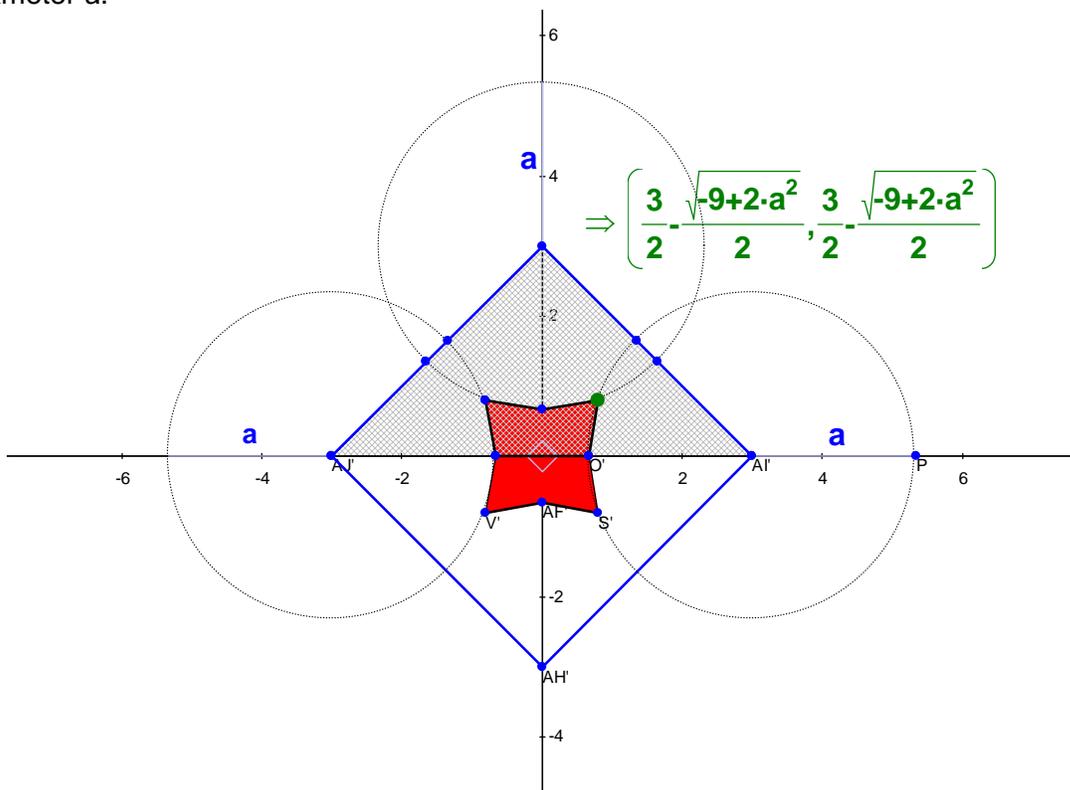
3.

3. Investigate the case c where $a > 3$. In this case concave quadrilateral shapes are formed at the vertices of the square with a convex quadrilateral at the origin. Students can again use circles and segments, construct the design and approximate the curved shapes with squares to look for specific ratio of areas for the design.



$$\Rightarrow \left( \frac{3}{2} - \frac{\sqrt{-9+2 \cdot a^2}}{2}, \frac{3}{2} - \frac{\sqrt{-9+2 \cdot a^2}}{2} \right)$$